Normalized boundary triplet for a sum of tensor products of operators

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The boundary triplet approach is applied to the construction of self-adjoint extensions of the operator having the form $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$ where the operator A is symmetric and the operator T is self-adjoint. A normalized boundary triplet is constructed, and formulas for the γ-field and the Weyl function are obtained. Applications to Schrödinger and Dirac operators in 1D are given.

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1. INTRODUCTION

In the following we are going to describe point interactions of a quantum system $\{\mathfrak{H}, A_0\}$ with a quantum reservoir $\{\mathfrak{T}, T\}$. By A_0 and T we denote self-adjoint operators acting on the separable Hilbert spaces $\mathfrak{H}$ and $\mathfrak{T}$, respectively. At first, let us recall the general philosophy of modeling point interactions in quantum mechanics. Let $\{\mathfrak{K}, S_0\}$ be a quantum system where S_0 is a self-adjoint operator acting on the separable Hilbert space $\mathfrak{K}$. To describe point interactions one restricts the self-adjoint operator S_0 to a densely defined closed symmetric operator S and extends it to another self-adjoint operator S'. The new self-adjoint operator S' is called the Hamiltonian of the perturbed system, that means, it takes into account the point interaction. Which extension one has to choose depends on the physical problem. Typical examples are δ and δ'-point interactions, cf.[4,5]. From the mathematical point of view it is interesting to note that the problem of describing point interactions fits into the framework of extension theory for symmetric operators.

To describe point interactions of a quantum system with a reservoir one has to specify the approach. At first, one considers the compound system consisting of the quantum system $\{\mathfrak{H}, A_0\}$ and the reservoir $\{\mathfrak{T}, T\}$, cf.[12]. Its Hamiltonian is given by the self-adjoint operator

$$S_0 := A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$$

where S_0 acts in the Hilbert space $\mathfrak{K} := \mathfrak{H} \otimes \mathfrak{T}$. To model the interaction with the quantum reservoir we act as follows. We restrict the self-adjoint operator A_0 to a densely defined closed symmetric operator A and consider the densely defined closed symmetric operator

$$S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T. \tag{1.1}$$

From the physical point of view, the restriction of A_0 to A and the following extension can be regarded as the opening of the quantum system $\{\mathfrak{H}, A_0\}$. To describe point interactions one has to extend the symmetric operator to a self-adjoint one which is different from S_0. However, not every self-adjoint extension S' of S, different from S_0, can be regarded as a Hamiltonian describing a point interaction with the reservoir. In fact, among them are extensions admitting the representation

$$S' = A' \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T \tag{1.2}$$

where A' is a self-adjoint extension of A. Of course, such Hamiltonians do not describe any interaction with the reservoir. From the physical point of view it is very important to describe all those extensions, which really describe point interactions with the reservoir. An example of this type can be found in[18].

An appropriate method to describe all self-adjoint extensions of a symmetric operator is the so-called boundary triplet approach, cf.[15–17,20,22,31]. A boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$, corresponding to a densely defined closed symmetric operator S^*, consists of an auxiliary Hilbert space $\mathcal{H}^S$ and linear maps $\Gamma_0^S, \Gamma_1^S : \mathrm{dom}\,(S^*) \longrightarrow \mathcal{H}^S$ such that the 'abstract Green's identity'

$$(S^* f, g) - (f, S^* g) = (\Gamma_1^S f, \Gamma_0^S g) - (\Gamma_0^S f, \Gamma_1^S g), \qquad f, g \in \mathrm{dom}\,(S^*),$$

is satisfied and the map

$$\Gamma^S := \begin{pmatrix} \Gamma_0^S \\ \Gamma_1^S \end{pmatrix} : \mathrm{dom}\,(S^*) \longrightarrow \begin{matrix} \mathcal{H}^S \\ \oplus \\ \mathcal{H}^S \end{matrix}$$

is surjective. Between the set of self-adjoint extensions of S and the set of self-adjoint relations in $\mathcal{H}^S$ there is a one-to-one correspondence. In other words, if S' is a self-adjoint extension of S, then there is a unique self-adjoint relation Θ' in $\mathcal{H}^S$ such that

$$\mathrm{dom}\,(S') = \mathrm{dom}\,(S_{\Theta'}) := \{f \in \mathrm{dom}\,(S^*) : \Gamma^S f \in \Theta'\}.$$

Conversely, if Θ' is a self-adjoint relation in $\mathcal{H}^S$, then $S_{\Theta'} := S^* \upharpoonright \mathrm{dom}\,(S_{\Theta'})$ defines a self-adjoint extension of S. Among all self-adjoint extensions there are two special ones: $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$ and $S_1 := S^* \upharpoonright \ker(\Gamma_1^S)$ which correspond to self-adjoint relations $\Theta_0 := \begin{pmatrix} 0 \\ f \end{pmatrix}$, and $\Theta_1 := \begin{pmatrix} f \\ 0 \end{pmatrix}$, $f \in \mathcal{H}$, respectively. Notice that $\Theta_1^{-1} = \Theta_0$.

For a given symmetric operator there are a lot of boundary triplets. In particular, having a boundary triplet one can easily construct a new one. It turns out, that if $\widetilde{S}$ is a given self-adjoint extension of S, then there is always a boundary triplet Π^S such that $S_0 := S^* \upharpoonright \ker(\Gamma_0^S) = \widetilde{S}$.

Important quantities, related to a boundary triplet, are the so-called Gamma field $\gamma^S(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1}$, $\mathfrak{N}_z := \ker(S^* - z)$, and the Weyl function $M^S(z) := \Gamma_1^S \gamma^S(z)$, $z \in \rho(S_0)$, respectively. Notice that the Krein-type resolvent formula

$$(S_\Theta - z)^{-1} - (S_0 - z)^{-1} = \gamma^S(z)(\Theta - M^S(z))^{-1}\gamma^S(\overline{z})^*, \quad z \in \rho(S_0) \cap \rho(\widetilde{S}),$$

holds for any self-adjoint relation Θ in $\mathcal{H}^S$.

Since any densely defined closed symmetric operator with equal deficiency indices admits a boundary triplet, we can find a boundary triplet for the symmetric operator S defined by (1.1). However, such an abstract boundary triplet suffers from the disadvantage that it is not clear whether the extension really describes a point interaction with the reservoir or the extension is of type (1.2). So we are going to find a more specific boundary triplet that distinguishes both types of extensions. To this end, we start with a given boundary triplet $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ for A^*. Outgoing from this boundary triplet for A^* we construct a boundary triplet $\Pi^S := \{\mathcal{H}^S, \Gamma_0^S, \Gamma^S\}$ for S^* which respects the tensor structure of the problem. That means, the boundary value space $\mathcal{H}^S$ should be given by $\mathcal{H}^S = \mathcal{H}^A \otimes \mathfrak{T}$ and the boundary maps Γ_0^S and Γ_1^S, roughly speaking, by

$$\Gamma_0^S := \Gamma_0^A \otimes I_{\mathfrak{T}} \quad \text{and} \quad \Gamma_1^S := \Gamma_1^A \otimes I_{\mathfrak{T}}. \tag{1.3}$$

This has the advantage that extensions of the structure (1.2) correspond to a self-adjoint relation Θ' of the form $\Theta' = \Theta'_A \otimes I_{\mathfrak{T}}$ where Θ'_A is the self-adjoint relation which corresponds to the self-adjoint extension A'. In other words, if the self-adjoint relation Θ' does not admit the tensor structure $\Theta' = \Theta'_A \otimes I_{\mathfrak{T}}$, then the corresponding self-adjoint extension $S_{\Theta'}$ can be regarded as a Hamiltonian which really describes a point interaction with a reservoir. However, this simple idea cannot be realized in such a straightforward manner. In particular, the boundary maps Γ_0^S and Γ_1^S are not well defined by (1.3). Moreover, additional difficulties arise from the fact that T is unbounded. The case where T is bounded was treated in[9] and is much easier. However, this case is usually not realized in physical applications.

Using the special boundary triplet from above, we apply it to describe point interactions of the Laplacian living on a bounded interval of the real axis and on the half-line with a photon reservoir and of the Dirac operator living on a bounded interval of the real axis and on the half-line with a photon reservoir.

Notation. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be separable Hilbert spaces. The set of closed (bounded) linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ is denoted by $\mathcal{C}(\mathfrak{H}_1, \mathfrak{H}_2)$ ($[\mathfrak{H}_1, \mathfrak{H}_2]$); $\mathcal{C}(\mathfrak{H}) = \mathcal{C}(\mathfrak{H}, \mathfrak{H})$, $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. By $\mathfrak{S}_p(\mathfrak{H})$, $p \in (0, \infty]$, we denote the Schatten-v.Neumann ideals of order p on $\mathfrak{H}$; in particular, $\mathfrak{S}_\infty(\mathfrak{H})$ denotes the ideal of compact operators on $\mathfrak{H}$.

By $\mathrm{dom}\,(T)$, $\mathrm{ran}\,(T)$ and $\rho(T)$ we denote the domain, range and resolvent set of the operator T, respectively.

2. PRELIMINARIES

A. Linear relations

A linear relation Θ in $\mathcal{H}$ is a closed linear subspace of $\mathcal{H} \oplus \mathcal{H}$. The set of all linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. Denote also by $\mathcal{C}(\mathcal{H})$ the set of all closed linear (not necessarily densely defined) operators in $\mathcal{H}$. Identifying each operator $T \in \mathcal{C}(\mathcal{H})$ with its graph $\mathrm{gr}\,(T)$ we regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$.

The role of the set $\widetilde{\mathcal{C}}(\mathcal{H})$ in extension theory becomes clear from Proposition 2.3. However, it's role in the operator theory is substantially motivated by the following circumstances: in contrast to $\mathcal{C}(\mathcal{H})$, the set $\widetilde{\mathcal{C}}(\mathcal{H})$ is closed with respect to taking inverse and adjoint relations Θ^{-1} and Θ^*. Here $\Theta^{-1} = \{\{g, f\} : \{f, g\} \in \Theta\}$ and

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called symmetric if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

B. Boundary triplets and proper extensions

Let us briefly recall some basic facts on boundary triplets. Let S be a densely defined closed symmetric operator with equal deficiency indices $0 < n_\pm(S) := \dim(\mathfrak{N}_{\pm \mathrm{i}})$, $\mathfrak{N}_z := \ker(S^* - z)$, $z \in \mathbb{C}_\pm$, acting on some separable Hilbert space $\mathfrak{H}$.

Definition 2.1 *A closed extension $\widetilde{S}$ of S is called proper if* $\mathrm{dom}\,(S) \subsetneqq \mathrm{dom}\,(\widetilde{S}) \subsetneqq \mathrm{dom}\,(S^*)$.

We denote by $\mathrm{Ext_S}$ the set of all proper extensions of S completed by the non-proper extensions S and S^*. For instance, any self-adjoint or maximal dissipative (accumulative) extension is proper.

Definition 2.2 (cf.[20]) A triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$, where $\mathcal{H}^S$ is an auxiliary Hilbert space and $\Gamma_0^S, \Gamma_1^S : \mathrm{dom}\,(S^*) \to \mathcal{H}^S$ are linear mappings, is called a boundary triplet for S^* if the 'abstract Green's identity'

$$(S^*f, g) - (f, S^*g) = (\Gamma_1^S f, \Gamma_0^S g) - (\Gamma_0^S f, \Gamma_1^S g), \quad f, g \in \mathrm{dom}\,(S^*), \tag{2.1}$$

is satisfied and the mapping $\Gamma^S := (\Gamma_0^S, \Gamma_1^S)^t : \mathrm{dom}\,(S^*) \to (\mathcal{H}^S \oplus \mathcal{H}^S)^t$ is surjective, i.e. $\mathrm{ran}\,(\Gamma^S) = (\mathcal{H}^S \oplus \mathcal{H}^S)^t$.

A boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ for S^* always exists whenever $n_+(S) = n_-(S)$. Note also that $n_\pm(S) = \dim(\mathcal{H}^{\mathrm{S}})$ and $\ker(\Gamma_0^S) \cap \ker(\Gamma_1^S) = \mathrm{dom}\,(S)$.

In general, the linear maps $\Gamma_j^S : \mathrm{dom}\,(S^*) \longrightarrow \mathcal{H}^S$, $j = 0, 1$, are neither bounded nor closed. However, equipping the domain $\mathrm{dom}\,(S^*)$ with the graph norm

$$\|f\|_{S^*}^2 := \|S^*f\|^2 + \|f\|^2, \qquad f \in \mathrm{dom}\,(S^*),$$

one gets a Hilbert space, which is denoted by $\mathfrak{H}_+(S^*)$, and regarding the maps $\Gamma_j^S : \mathrm{dom}\,(S^*) \longrightarrow \mathcal{H}^S$, $j = 0, 1$, as acting from $\mathfrak{H}_+(S^*)$ into $\mathcal{H}^S$ it turns out that the operators $\Gamma_j^S : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}^S$, $j = 0, 1$, are bounded. In the following we denote the operator $\Gamma_j^S : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}^S$ by $\widehat{\Gamma}_j^S : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}^S$, $j = 0, 1$. If $J_{S^*} : \mathfrak{H}_+(S^*) \longrightarrow \mathrm{dom}\,(S^*)$ denotes the embedding operator, then we have $\widehat{\Gamma}_j^S = \Gamma_j^S J_{S^*}$, $j = 0, 1$. From the surjectivity it follows that $\mathrm{ran}\,(\widehat{\Gamma}^S) = \mathcal{H}^S \oplus \mathcal{H}^S$, where $\widehat{\Gamma}^S := (\widehat{\Gamma}_0^S, \widehat{\Gamma}_1^S)^t$. Notice that the abstract Green's identity (2.1) can be written as

$$(S^* J_{S^*} f, J_{S^*} g) - (J_{S^*} f, S^* J_{S^*} g) = (\widehat{\Gamma}_1^S f, \widehat{\Gamma}_0^S g) - (\widehat{\Gamma}_0^S f, \widehat{\Gamma}_1^S g), \quad f, g \in \mathfrak{H}_+(S^*).$$

With any boundary triplet Π_S one associates two canonical self-adjoint extensions $S_j := S^* \upharpoonright \ker(\Gamma_j^S)$, $j \in \{0, 1\}$. Conversely, for any extension $S_0 = S_0^* \in \mathrm{Ext_S}$ there exists a (non-unique) boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ for S^* such that $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$.

Using the concept of boundary triplets one can parameterize all proper extensions of A in the following way.

Proposition 2.3 (cf.[16,25]) *Let $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for S^*. Then the mapping*

$$\mathrm{Ext_S} \ni \widetilde{\mathrm{S}} \to \Gamma^{\mathrm{S}}\mathrm{dom}\,(\widetilde{\mathrm{S}}) = \{(\Gamma_0^{\mathrm{S}}\mathrm{f}, \Gamma_1^{\mathrm{S}}\mathrm{f})^{\mathrm{t}} :\ \mathrm{f} \in \mathrm{dom}\,(\widetilde{\mathrm{S}})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}^{\mathrm{S}}) \tag{2.2}$$

establishes a bijective correspondence between the sets $\mathrm{Ext_S}$ *and* $\widetilde{\mathcal{C}}(\mathcal{H}^S)$. *We write* $\widetilde{S} = S_\Theta$ *if* $\widetilde{S}$ *corresponds to* Θ *by* (2.2). *Moreover, the following holds:*

(i) $S_\Theta^* = S_{\Theta^*}$, *in particular,* $S_\Theta^* = S_\Theta$ *if and only if* $\Theta^* = \Theta$.

(ii) S_Θ *is symmetric (self-adjoint) if and only if* Θ *is symmetric (self-adjoint).*

In particular, $S_j := S^* \upharpoonright \ker(\Gamma_j^S) = S_{\Theta_j}$, $j \in \{0, 1\}$, where $\Theta_0 := \begin{pmatrix} \{0\} \\ \mathcal{H}^S \end{pmatrix}$ and $\Theta_1 := \begin{pmatrix} \mathcal{H}^S \\ \{0\} \end{pmatrix} = \mathrm{gr}\,(\mathbb{O})$ where $\mathbb{O}$ denotes the zero operator in $\mathcal{H}^S$. Note also that $\widetilde{\mathcal{C}}(\mathcal{H}^S)$ contains the trivial linear relations $\{0\} \times \{0\}$ and $\mathcal{H}^S \times \mathcal{H}^S$ parameterizing the extensions S and S^*, respectively, for any boundary triplet Π_S.

C. Gamma field and Weyl function

It is well known that the Weyl function is an important tool in the direct and inverse spectral theory of Sturm-Liouville operators. In[15,16] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator S with $n_+(S) = n_-(S) \leq \infty$. Following[16], we briefly recall basic facts on Weyl functions and γ-fields, associated with a boundary triplet Π. For further properties and applications see[10,16,17],[31] (and references therein).

Definition 2.4 (cf.[15,16]) Let $\Pi^S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for S^* and $S_0 = S^* \upharpoonright \ker(\Gamma_0^S)$. The operator valued functions $\gamma^S(\cdot) : \rho(S_0) \to [\mathcal{H}^S, \mathfrak{H}]$ and $M^S(\cdot) :\ \rho(S_0) \to [\mathcal{H}^S]$ defined by

$$\gamma^S(z) := \left(\Gamma_0^S \upharpoonright \mathfrak{N}_z\right)^{-1} \qquad \text{and} \qquad M^S(z) := \Gamma_1^S \gamma^S(z), \quad z \in \rho(S_0), \tag{2.3}$$

are called the γ-field and the Weyl function, respectively, corresponding to the boundary triplet Π_S.

Clearly, the Weyl function can equivalently be defined by

$$M^S(z)\Gamma_0^S f_z = \Gamma_1^S f_z, \qquad f_z \in \mathfrak{N}_z, \quad z \in \rho(S_0).$$

The γ-field $\gamma^S(\cdot)$ and the Weyl function $M^S(\cdot)$ in (2.3) are well defined. Moreover, both $\gamma^S(\cdot)$ and $M^S(\cdot)$ are holomorphic on $\rho(S_0)$ and satisfy the following relations

$$\gamma^S(z) = \big(I + (z-\zeta)(S_0 - z)^{-1}\big)\gamma^S(\zeta), \qquad z, \zeta \in \rho(S_0), \tag{2.4}$$

and

$$M^S(z) - M^S(\zeta)^* = (z - \overline{\zeta})\gamma^S(\zeta)^*\gamma^S(z), \qquad z, \zeta \in \rho(S_0), \tag{2.5}$$

hold. Identity (2.5) yields that $M^S(\cdot)$ is an $[\mathcal{H}^S]$-valued Nevanlinna function ($M^S(\cdot) \in R[\mathcal{H}^S]$), i.e. $M^S(\cdot)$ is an $[\mathcal{H}^S]$-valued holomorphic function on $\mathbb{C}_\pm$ satisfying

$$M^S(z) = M^S(\overline{z})^* \qquad \text{and} \qquad \frac{\mathrm{Im}(M^S(z))}{\mathrm{Im}(z)} \geq 0, \qquad z \in \mathbb{C}_\pm.$$

It follows also from (2.5) that $0 \in \rho(\mathrm{Im}(M^S(z)))$ for all $z \in \mathbb{C}_\pm$.

Being an $R[\mathcal{H}^S]$-function the Weyl function $M^S(\cdot)$ admits an integral representation

$$M^S(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Sigma_S(t), \qquad \int_{\mathbb{R}} \frac{d\Sigma_S(t)}{1+t^2} \in [\mathcal{H}^S], \tag{2.6}$$

where $C_0 = C_0^*$ and $\Sigma_S(\cdot)$ is a left continuous ($\Sigma_S(t) = \Sigma_S(t-0)$) monotone operator-valued function. Emphasize that the linear term $C_1 z$ is missing in (2.6) because the operator A is densely defined (see[16]).

A Weyl function $M^S(\cdot)$ is said to be of scalar type if there exists a scalar Nevanlinna function $m^S(\cdot)$ such that the representation

$$M^S(z) = m^S(z) I_{\mathcal{H}^S}, \quad z \in \mathbb{C}_\pm,$$

holds where $I_{\mathcal{H}^S}$ is the identity operator in $\mathcal{H}^S$, see[3]. Obviously, $M^S(\cdot)$ is of scalar type if $n_\pm(A) = 1$.

Next we extract from (2.6) lower and upper bounds for $\mathrm{Im}(M^S(i-\lambda))$ which will be useful in the sequel. It follows from (2.6) that

$$\mathrm{Im}\big(M^S(i-\lambda)\big) = \int_{\mathbb{R}} \frac{d\Sigma_S(t)}{(t-\lambda)^2+1}, \quad \lambda \in \mathbb{R} \tag{2.7}$$

Note that with certain positive constants $C_1, C_2 > 0$ the following estimate holds

$$\frac{C_1}{1+|\lambda|^2} \leq \frac{1+t^2}{(t-\lambda)^2+1} \leq C_2(1+|\lambda|^2), \qquad \lambda \in \mathbb{R}.$$

Combining these estimates with the identity $\mathrm{Im}M(i) = \int_{\mathbb{R}}(1+t^2)^{-1}d\Sigma_S(t)$ one derives from (2.7) that

$$C_1(1+|\lambda|^2)^{-1}\mathrm{Im}M(i) \leq \mathrm{Im}M^S(i-\lambda) \leq C_2(1+|\lambda|^2)\mathrm{Im}M(i), \qquad \lambda \in \mathbb{R}. \tag{2.8}$$

Emphasize that since the proof of estimates (2.8) is based only on integral representation (2.6), these estimates are valid for any $R[\mathcal{H}^S]$-function not necessarily being a Weyl function.

D. Krein-type formula for resolvents

Let $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for S^*, and $M^S(\cdot)$ and $\gamma^S(\cdot)$ the corresponding Weyl function and γ-field, respectively. For any proper (not necessarily self-adjoint) extension $\widetilde{S}_\Theta \in \mathrm{Ext_S}$ with non-empty resolvent set $\rho(\widetilde{S}_\Theta)$ the following Krein-type formula holds (cf.[15–17])

$$(S_\Theta - z)^{-1} - (S_0 - z)^{-1} = \gamma^S(z)(\Theta - M^S(z))^{-1}(\gamma^S(\overline{z}))^*, \quad z \in \rho(S_0) \cap \rho(S_\Theta). \tag{2.9}$$

Formula (2.9) extends the known Krein formula for canonical resolvents to the case of any $S_\Theta \in \mathrm{Ext_S}$ with $\rho(S_\Theta) \neq \emptyset$. Moreover, due to relations (2.2) and (2.3) all objects in formula (2.9) are expressed by means of the boundary triplet Π_S. We emphasize, that this connection makes it possible to apply the Krein-type formula (2.9) to boundary value problems.

E. Normalized boundary triplets

Let S_n be a densely defined closed symmetric operator in $\mathfrak{H}_n$, $n \in \mathbb{Z}$, and let $S := \oplus_{n\in\mathbb{Z}} S_n$. Clearly,

$$S^* = \bigoplus_{n\in\mathbb{Z}} S_n^*, \qquad \mathrm{dom}\,(S^*) = \left\{ f = \oplus_{\mathbb{Z}} f_n \in \mathfrak{H} : \ f_n \in \mathrm{dom}\,(S_n^*), \ \sum_{n=1}^{\infty} \|S_n^* f_n\|^2 < \infty \right\}. \tag{2.10}$$

Let $\Pi_{S_n} = \{\mathcal{H}^{S_n}, \Gamma_0^{S_n}, \Gamma_1^{S_n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{Z}$. Define mappings Γ_0^S and Γ_1^S by setting

$$\Gamma_j^S := \bigoplus_{n\in\mathbb{Z}} \Gamma_j^{S_n}, \qquad \mathrm{dom}\,(\Gamma_j^S) := \left\{ \oplus_{n\in\mathbb{Z}} f_n \in \mathrm{dom}\,(S^*) : \sum_{n\in\mathbb{Z}} \|\Gamma_j^{S_n} f_n\|^2 < \infty \right\}, \quad j \in \{0,1\}. \tag{2.11}$$

Definition 2.5 *Let Γ_j^S be defined by (2.11) and $\mathcal{H}^S := \oplus_{n\in\mathbb{Z}} \mathcal{H}^{S_n}$. A collection $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ will be called a* direct sum of boundary triplets *and will be assigned as $\Pi_S = \bigoplus_{n\in\mathbb{Z}} \Pi_{S_n}$.*

It was first discovered by A. Kochubei[22] that the direct sum $\oplus \Pi_n$ of boundary triplets Π_n is not a boundary triplet in general. Later on simple examples were constructed in[26],[23],[11]. Moreover, it was shown in[23] (Theorem 3.2) that Π_S is only a generalized boundary triplet (a boundary relation in the sense of[14]). Moreover, according to[13] Π_S is a so called ES-generalized boundary triplet for S^*, since the operator $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$ is essentially self-adjoint.

The reason is that the domain $\mathrm{dom}\,(\Gamma_j^S)$, $j \in \{0,1\}$, might be narrower than $\mathrm{dom}\,(S^*)$ and the range of the mapping $\Gamma^S := (\Gamma_0^S, \Gamma_1^S)^t : \mathrm{dom}\,(S^*) \to (\mathcal{H}^S \oplus \mathcal{H}^S)^t$ might be a proper subset of $(\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Nevertheless, $\mathrm{dom}\,(\Gamma_j^S)$, $j \in \{0,1\}$, is always dense in $\mathfrak{H}_+(S^*)$ and its range $\mathrm{ran}\,(\Gamma^S)$ is dense in $(\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Moreover, by[14] (Propostion 5.3), Π_S is a boundary triplet whenever $\mathrm{ran}\,(\Gamma^S) = (\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Besides, in accordance with[23] (Proposition 3.8) the conditions

$$\sum_{n\in\mathbb{Z}} \|\Gamma_j^{S_n} f_n\|^2 < \infty, \quad f = \oplus_{n\in\mathbb{Z}} f_n \in \mathrm{dom}\,(S^*), \quad j \in \{0,1\}, \tag{2.12}$$

imply that $\Pi_S = \bigoplus_{n\in\mathbb{Z}} \Pi_{S_n}$ is an ordinary boundary triplet, while the sole first condition in (2.12) (with $j = 0$) ensures only that Π_S is a B-generalized boundary triplet in the sense of[17],[13]. Moreover, according to[13] Π_S is a so called ES-generalized boundary triplet for S^*, since the operator $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$ is essentially self-adjoint.

A regularization procedure described below was first proposed in[26] and has been applied to construct a boundary triplet for Sturm-Liouville operators

$$-d^2/dx^2 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T, \qquad \mathfrak{H} = L^2(\mathbb{R}_+; \mathfrak{T}) = L^2(\mathbb{R}_+) \otimes \mathfrak{T}, \tag{2.13}$$

with unbounded potential $T = T^* \in \mathcal{C}(\mathfrak{T})$. Further generalizations of regularization procedures as well as applications to Schrödinger and Dirac operators with δ-interactions were obtained in[23] and[11], respectively.

Let $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for S^* with Weyl function $M^S(\cdot)$. We call Π_S a normalized boundary triplet for S^* if the condition $M^S(i) = iI_{\mathcal{H}^S}$ is satisfied.

Lemma 2.6 ([26]) *Let $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for S^*, let $\gamma^S(\cdot)$ and $M^S(\cdot)$ be the $\gamma(\cdot)$-field and Weyl function, respectively. Let $R_S := \sqrt{\mathrm{Im}(M^S(i))}$ and $Q_S := \mathrm{Re}(M^S(i))$. Then $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$, where*

$$\widetilde{\mathcal{H}}^S := \mathcal{H}^S, \quad \widetilde{\Gamma}_0^S := R_S \Gamma_0^S \quad \text{and} \quad \widetilde{\Gamma}_1^S := R_S^{-1}(\Gamma_1^S - Q_S \Gamma_0^S), \tag{2.14}$$

is a normalized boundary triplet for S^* such that

$$S_0 := S^* \upharpoonright \ker(\Gamma_0^S) = S^* \upharpoonright \ker(\widetilde{\Gamma}_0^S).$$

The γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ corresponding to the triplet $\widetilde{\Pi}_S$ are given by

$$\widetilde{\gamma}^S(z) = \gamma^S(z) R_S^{-1} \quad \text{and} \quad \widetilde{M}^S(z) = R_S^{-1}(M^S(z) - Q_S) R_S^{-1}, \qquad z \in \mathbb{C}_\pm. \tag{2.15}$$

Lemma 2.6 shows that with any boundary triplet one can associate a normalized boundary triplet such that S_0 remains unchanged. The following theorem presents a regularization procedure for direct sums $\Pi_S = \bigoplus_{n\in\mathbb{Z}} \Pi_{S_n}$ to define an ordinary boundary triplet.

Theorem 2.7 (Theorem 3.3, [26]**)** *Let S_n be a densely defined closed symmetric operator in $\mathfrak{H}_n$, $n \in \mathbb{Z}$, and $S := \oplus_{n\in\mathbb{Z}} S_n$. Let $\Pi_{S_n} = \{\mathcal{H}^{s_n}, \Gamma_0^{S_n}, \Gamma_1^{S_n}\}$ be a boundary triplet for S_n^*, $S_{0n} := S_n^* \upharpoonright \ker(\Gamma_0^{S_n})$, $n \in \mathbb{Z}$, and let $\gamma^{S_n}(\cdot)$ and $M^{S_n}(\cdot)$ be the corresponding γ-field and Weyl function, respectively. Finally, let $R_{S_n} := \sqrt{\mathrm{Im}(M^{S_n}(i))}$ and $Q_{S_n} := \mathrm{Re}(M^{S_n}(i))$, $n \in \mathbb{Z}$. Then the triplet $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ with*

$$\widetilde{\mathcal{H}}^S := \bigoplus_{n\in\mathbb{N}} \mathcal{H}^{S_n}, \quad \widetilde{\Gamma}_0^S := \bigoplus_{n\in\mathbb{Z}} R_{S_n}\Gamma_0^{S_n}, \quad \widetilde{\Gamma}_1^S := \bigoplus_{n\in\mathbb{Z}} R_{S_n}^{-1}\left(\Gamma_1^{S_n} - Q_{S_n}\Gamma_0^{S_n}\right), \tag{2.16}$$

is a (normalized) boundary triplet for S^* satisfying

$$\widetilde{S}_0 = S^* \upharpoonright \ker(\widetilde{\Gamma}_0^S) = \bigoplus_{n\in\mathbb{Z}} \widetilde{S}_{0n} = \bigoplus_{n\in\mathbb{Z}} S_{0n}, \qquad \widetilde{S}_{0n} = S_n^* \upharpoonright \ker(\widetilde{\Gamma}_0^{S_n}). \tag{2.17}$$

Moreover, the γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ corresponding to $\widetilde{\Pi}_S$ are given by

$$\widetilde{\gamma}^S(z) = \bigoplus_{n\in\mathbb{Z}} \gamma^{S_n}(z) R_{S_n}^{-1} \quad \textit{and} \quad \widetilde{M}^S(z) = \bigoplus_{n\in\mathbb{Z}} R_{S_n}^{-1}\left(M^{S_n}(z) - Q_{S_n}\right) R_{S_n}^{-1}, \quad z \in \mathbb{C}_\pm. \tag{2.18}$$

Next we assume that the operator $S = \bigoplus_{n=1}^\infty S_n$ has a regular real point, i.e., there exists $a = \overline{a} \in \hat{\rho}(A)$. The latter is equivalent to the existence of $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset \cap_{n=1}^\infty \widehat{\rho}(S_n). \tag{2.19}$$

Emphasize that condition $a \in \cap_{n=1}^\infty \widehat{\rho}(S_n)$ is not sufficient for the inclusion $a \in \widehat{\rho}(A)$.

It is known (see e.g.[24],[1]) that under condition (2.19) for every $k \in \mathbb{N}$ there exists a selfadjoint extension $\widetilde{S}_k = \widetilde{S}_k^*$ of S_k preserving the gap $(a - \varepsilon, a + \varepsilon)$. The latter amounts to saying that the Weyl function of the pair $\{S_k, \widetilde{S}_k\}$ is regular within the gap $(a - \varepsilon, a + \varepsilon)$.

For operators $S = \bigoplus_{n=1}^\infty S_n$ satisfying (2.19) we complete Theorem 2.7 by presenting a regularization procedure for $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ leading to a boundary triplet (cf.[23] (Theorem 3.13),[11] (Theorem 2.12 and Corollary 2.13)). In applications to symmetric operators with a gap this regularization is more appropriate and simpler than the one described in Theorem 2.7.

Proposition 2.8 ([11,23]) *Let $\{S_n\}_{n=1}^\infty$ be a sequence of symmetric operators satisfying* (2.19). *Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\Gamma_0^{(n)})$. Let also $\gamma^{S_n}(\cdot)$ and $M_n(\cdot) := M^{S_n}(\cdot)$ be the corresponding γ-field and Weyl function, respectively. Assume also that for some operators R_n such that $R_n, R_n^{-1} \in [\mathcal{H}_n]$, the following conditions are satisfied*

$$\sup_n \|R_n^{-1}(M_n'(a))(R_n^{-1})^*\|_{\mathcal{H}_n} < \infty \quad \textit{and} \quad \sup_n \|R_n^*(M_n'(a))^{-1}R_n\|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{N}. \tag{2.20}$$

Then the direct sum $\widetilde{\Pi}_S = \bigoplus_{n=1}^\infty \widetilde{\Pi}_n$ of boundary triplets where

$$\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\} \quad \textit{with} \quad \widetilde{\Gamma}_0^{(n)} := R_n\Gamma_0^{(n)}, \qquad \widetilde{\Gamma}_1^{(n)} := (R_n^{-1})^*\big(\Gamma_1^{(n)} - M_n(a)\Gamma_0^{(n)}\big), \tag{2.21}$$

forms a boundary triplet for $S^* = \bigoplus_{n=1}^\infty S_n^*$.

Moreover, the corresponding γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ are given by

$$\widetilde{\gamma}^S(z) = \bigoplus_{n\in\mathbb{Z}} \gamma^{S_n}(z) R_n^{-1} \quad \textit{and} \quad \widetilde{M}^S(z) = \bigoplus_{n\in\mathbb{Z}} (R_n^{-1})^* \left(M_n(z) - M_n(a)\right) R_n^{-1}, \quad z \in \mathbb{C}_\pm. \tag{2.22}$$

In particular one can set $R_n = \sqrt{M_n'(a)}$, $n \in \mathbb{N}$.

Emphasize that $M_n'(a)$ is a positive definite operator whenever $a \in \rho(S_{n0})$.

3. OPERATOR-SPECTRAL INTEGRALS

Let $F(\cdot)$ be an orthogonal operator measure with compact support $\mathrm{supp}\,(F) \subseteq \Delta := [a, b)$, $-\infty < a < b < \infty$, and with values in $[\mathcal{H}]$. Further, let $\Omega(\cdot) : [a, b) \longrightarrow [\mathcal{H}, \mathcal{H}_1]$ be an operator-valued function. We consider partitions $\mathcal{Z}$ of

$[a,b)$ of the form $[a,b) = [\lambda_0,\lambda_1)\cup[\lambda_1,\lambda_2)\cup\ldots\cup[\lambda_{n-1},\lambda_n)$, $\lambda_0 = a$, $\lambda_n = b$ and set $\Delta_m := [\lambda_{m-1},\lambda_m)$, $m = 1,2,\ldots,n$. Thus $[a,b) = \bigcup_{m=1}^n \Delta_m$ and the Δ_m are pairwise disjoint. Let $|\mathcal{Z}| := \max_{m=1,2,\ldots,n} |\Delta_m|$, where $|\Delta_m| := \lambda_m - \lambda_{m-1}$. We define the operator $\Sigma_{\mathcal{Z}}\Omega$ by

$$\Sigma_{\mathcal{Z}}\Omega = \sum_{m=1}^{n} \Omega(x_m)F(\Delta_m), \qquad x_m \in \Delta_m.$$

The sum $\Sigma_{\mathcal{Z}}\Omega$ is called the Riemann-Stieltjes sum of $\Omega(\cdot)$ with respect to the operator measure $F(\cdot)$. If there is an operator $\Sigma_0 \in [\mathcal{H},\mathcal{H}_1]$ such that $\lim_{|\mathcal{Z}|\to 0} \|\Sigma_{\mathcal{Z}}\Omega - \Sigma_0\| = 0$ independent of the special choice of $\mathcal{Z}$ and $\{x_m\}_{m=1}^n$, then Σ_0 is called the operator spectral integral of $\Omega(\cdot)$ with respect to $F(\cdot)$ and is denoted by

$$\Sigma_0 =: \int_\Delta \Omega(\lambda)F(d\lambda). \tag{3.1}$$

Obviously, in a similar way one can define for operator-valued functions $\Omega : \Delta \longrightarrow [\mathcal{H}_1,\mathcal{H}]$ the operator spectral integral $\int_\Delta F(d\lambda)\Omega(\lambda)$ as the limit of the Riemann-Stieltjes sums $\sum_m F(\Delta_m)\Omega(x_m)$. It is clear that the operator spectral integral is linear with respect to $\Omega(\cdot)$. If B is a bounded operator, then

$$B\int_\Delta \Omega(\lambda)F(d\lambda) = \int_\Delta B\Omega(\lambda)F(d\lambda).$$

Definition 3.1 *The operator-valued mapping $\Omega : [a,b) \longrightarrow [\mathcal{H}]$ will be called F-admissible, if the integral $\int_\Delta \Omega(\lambda)F(d\lambda)$ exists and*

$$\Omega(\lambda)F(\delta) = F(\delta)\Omega(\lambda)F(\delta), \qquad \delta \in \mathcal{B}([a,b)), \quad \lambda \in \Delta. \tag{3.2}$$

Proposition 3.2 *Let $\Omega : [a,b) \longrightarrow [\mathcal{H}]$ be F-admissible, $\Omega_1 : [a,b) \longrightarrow [\mathcal{H},\mathcal{H}_1]$, and assume that $\int_\Delta \Omega_1(\lambda)F(d\lambda)$ exists. Then $\int_\Delta \Omega_1(\lambda)\Omega(\lambda)F(d\lambda)$ exists and*

$$\int_\Delta \Omega_1(\lambda)\Omega(\lambda)F(d\lambda) = \int_\Delta \Omega_1(\delta)F(d\delta)\int_\Delta \Omega(\mu)F(d\mu).$$

Proof. It is easily seen that

$$\Sigma_{\mathcal{Z}}\Omega_1\Sigma_{\mathcal{Z}}\Omega \longrightarrow \int_\Delta \Omega_1(\lambda)F(d\lambda)\int_\Delta \Omega(\mu)F(d\mu) \qquad \text{as} \qquad |\mathcal{Z}| \longrightarrow 0.$$

On the other hand, since the measure $F(\cdot)$ is orthogonal, $F(\Delta_j)F(\Delta_k) = F(\Delta_j)\delta_{jk}$, $j,k \in \{1,\ldots,m\}$. Combining these relations with the F-admissibility of Ω yields

$$\Sigma_{\mathcal{Z}}\Omega_1\Sigma_{\mathcal{Z}}\Omega = \sum_{m,m'=1}^{n} \Omega_1(x_m)F(\Delta_m)\Omega(x_{m'})F(\Delta_{m'}) = \sum_{m,m'=1}^{n} \Omega_1(x_m)F(\Delta_m)F(\Delta_{m'})\Omega(x_{m'})F(\Delta_{m'})$$

$$= \sum_{m=1}^{n} \Omega_1(x_m)F(\Delta_m)\Omega(x_m)F(\Delta_m) = \sum_{m=1}^{n} \Omega_1(x_m)\Omega(x_m)F(\Delta_m) \longrightarrow \int_\Delta \Omega_1(\lambda)\Omega(\lambda)F(d\lambda) \quad \text{as} \quad |\mathcal{Z}| \longrightarrow 0.$$

Combining both relations completes the proof. □

In what follows we assume that $\mathcal{H} = \mathcal{H}_1$.

Proposition 3.3 *Let $X : [a,b) \longrightarrow [\mathcal{H}]$ be an F-admissible function, and assume, in addition, that there exist real numbers c_1, c_2, such that $X(\lambda)$ is self-adjoint and $c_1 \le X(\lambda) \le c_2$, $\lambda \in \Delta$. Let $\varphi \in C[c_1,c_2]$. Then the following holds:*

(i) *The operator $\widehat{X} := \int_\Delta X(\lambda)F(d\lambda)$ is self-adjoint and satisfies $c_1 \le \widehat{X} \le c_2$;*

(ii) *The following estimate holds $\| \varphi(\widehat{X}) \| \le \| \varphi \|_\infty$*

(iii) *The operator-valued function $\varphi(X(\cdot))$ is F-admissible and*

$$\int_\Delta \varphi(X(\lambda))F(d\lambda) = \varphi(\widehat{X}). \tag{3.3}$$

Proof. (i) Let $\mathcal{Z}$ be any partition as above. Then for any $h \in \mathcal{H}$ one gets

$$\langle \Sigma_{\mathcal{Z}} X h, h\rangle = \sum_{m=1}^{n} \langle F(\Delta_m) X(x_m) F(\Delta_m) h, h\rangle \geq \sum_{m=1}^{n} c_1 \parallel F(\Delta_m) h \parallel^2 .$$

Thus $\langle \Sigma_{\mathcal{Z}} h, h\rangle \in \mathbb{R}$ and $\langle \Sigma_{\mathcal{Z}} h, h\rangle \geq c_1 \parallel h \parallel^2$. In the same way one shows that $\langle \Sigma_{\mathcal{Z}} h, h\rangle \leq c_2 \parallel h \parallel^2$. By passing to the limit, as $|\mathcal{Z}| \longrightarrow 0$, we get that $\langle \hat{X} h, h\rangle \in [c_1 \parallel h \parallel^2, c_2 \parallel h \parallel^2]$ for every $h \in \mathcal{H}$, and the assertion (i) is proved.

(ii) By the functional calculus, both inequalities $\parallel \varphi(\widehat{X}) \parallel \leq \parallel \varphi \parallel_\infty$ and $\parallel \varphi(X(\lambda)) \parallel \leq \parallel \varphi \parallel_\infty$ hold for every $\lambda \in \Delta$ and each continuous function $\varphi \in C[c_1, c_2]$.

(iii) First we prove, by induction, the assertion (iii) in the special case, when $\varphi(\lambda) = \lambda^n$. By the assumption, the assertion is true for $n = 1$. Suppose that it is true for $n = k$. Let us prove it for $n = k + 1$. One has

$$X^{k+1}(\lambda) F(\delta) = X(\lambda) F(\delta) X^k(\lambda) F(\delta) = F(\delta) X(\lambda) F(\delta) \cdot F(\delta) X^k(\lambda) F(\delta) = F(\delta) X^{k+1}(\lambda) F(\delta), \qquad \lambda \in \Delta, \delta \in \mathcal{B}(\Delta). \tag{3.4}$$

Therefore Proposition 3.2 ensures that the integral $\int_\Delta X^{k+1}(\lambda) F(d\lambda)$ exists and

$$\int_\Delta X^{k+1}(\lambda) F(d\lambda) = \widehat{X}^{k+1}.$$

By linearity, these equalities are easily extended for polynomials in λ.

Let φ be a continuous function, $\varphi \in C[c_1, c_2]$. By the Weierstrass Theorem, there exists a sequence $\{p_k\}_1^\infty$ of polynomials approaching φ in $C[c_1, c_2]$. In accordance with the functional calculus for selfadjoint operators,

$$\parallel \varphi(\widehat{X}) - p_k(\widehat{X}) \parallel \leq \parallel \varphi - p_k \parallel_\infty \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

and

$$\parallel \varphi(X(\lambda)) - p_k(X(\lambda)) \parallel \leq \parallel \varphi - p_k \parallel_\infty \longrightarrow 0 \quad \text{as} \quad k \to \infty, \qquad \lambda \in \Delta.$$

Combining this relation with equalities (3.4) for polynomials we obtain that for every $\lambda \in \Delta$ and any Borel subset $\delta \subset \Delta$ the following holds

$$\varphi(X(\lambda)) F(\delta) = \lim_{k \longrightarrow \infty} p_k(X(\lambda)) F(\delta) = \lim_{k \longrightarrow \infty} F(\delta) p_k(X(\lambda)) F(\delta) = F(\delta) \varphi(X(\lambda)) F(\delta), \quad \lambda \in \Delta.$$

This relation means that the function $\varphi(X(\cdot))$ satisfies commutation relation (3.2). To prove its F-admissibility it remains to prove the existence of the integral $\int_\Delta \varphi(X(\lambda)) F(d\lambda)$. We prove it together with relation (3.3). To this end for each partition $\mathcal{Z}$ of $[a, b]$ we prove the following estimate

$$\parallel \Sigma_{\mathcal{Z}}(\varphi(X) - p_k(X)) \parallel \leq \parallel \varphi - p_k \parallel_\infty .$$

Since the measure F is orthogonal, one gets

$$\begin{aligned} \parallel \Sigma_{\mathcal{Z}} \varphi(X) f - \Sigma_{\mathcal{Z}} p_k(X) f \parallel^2 &= \left\| \sum_{m=1}^{n} (\varphi - p_k)(X)(x_m) F(\Delta_m) f \right\|^2 \\ &= \sum_{m=1}^{n} \parallel (\varphi - p_k)(X)(x_m) F(\Delta_m) f \parallel^2 \\ &\leq \sum_{m=1}^{n} \parallel \varphi - p_k \parallel_\infty^2 \parallel F(\Delta_m) f \parallel^2 = \parallel \varphi - p_k \parallel_\infty^2 \parallel f \parallel^2 . \end{aligned}$$

Let $\{\mathcal{Z}_j\}$ be a sequence of partitions satisfying $|\mathcal{Z}_j| \longrightarrow 0$ and let $\varepsilon > 0$. Choose k such that $\parallel \varphi - p_k \parallel_\infty < \varepsilon$ and j_0 such that

$$\parallel \Sigma_{\mathcal{Z}_j} p_k(X) - \Sigma_{\mathcal{Z}_{j'}} p_k(X) \parallel < \varepsilon, \qquad j, j' \geq j_0,$$

and hence $\parallel \Sigma_{\mathcal{Z}_j} \varphi(X) - \Sigma_{\mathcal{Z}_{j'}} \varphi(X) \parallel < 3\varepsilon$ for all $j, j' \geq j_0$. Thus the limit $\lim_{j \longrightarrow \infty} \Sigma_{\mathcal{Z}_j} \varphi(X)$ exists, and

$$\parallel \lim_{j \longrightarrow \infty} \Sigma_{\mathcal{Z}_j} \varphi(X) - \varphi(\widehat{X}) \parallel \leq \parallel \lim_{j \longrightarrow \infty} \Sigma_{\mathcal{Z}_j}(\varphi(X) - p_k(X)) \parallel + \parallel p_k(\widehat{X}) - \varphi(\widehat{X}) \parallel < 2\varepsilon. \tag{3.5}$$

Since $\varepsilon > 0$ is arbitrary, this inequality ensures the existence of the integral $\int_\Delta \varphi(X(\lambda)) F(d\lambda)$, thus proves F-admissibility of $\varphi(X(\cdot))$. Moreover, estimate (3.5) proves equality (3.3). □

Denote by dm the Lebesgue measure on $\mathbb{R}$.

Corollary 3.4 *Assume that $\Omega(\cdot) = \Omega(\cdot)^*$ is a self-adjoint $[\mathcal{H}]$-valued Lipschitz function in $\Delta = [a, b)$ and $c_1 \le \Omega(\cdot) \le c_2$. Assume also that $F(\cdot)$ is a spectral measure in $\mathcal{H}$ with compact support, $\operatorname{supp}(F) \subseteq \Delta := [a, b)$, and $\varphi \in C[c_1, c_2]$. If in addition, commutation relation (3.2) holds, then the operator-valued function $\varphi(\Omega(\cdot))$ is F-admissible and*

$$\int_\Delta \varphi(\Omega(\lambda))F(d\lambda) = \varphi(\widehat{X}). \tag{3.6}$$

Proof. It is shown in[2] (Lemma 7.2) that the integral (3.1) exists whenever $\Omega(\cdot)$ is Lipschitz function. By Proposition 3.3(iii) $\varphi(\Omega(\cdot))$ is F-admissible and equality (3.6) holds. □

Corollary 3.5 *Let $\Omega(\cdot) = \Omega(\cdot)^*$ be differentiable with respect to the operator norm m-almost everywhere in $\Delta = [a, b)$, $c_1 \le \Omega(\cdot) \le c_2$, and let $\Omega(\cdot)$ be expressed by means of its derivative $\Omega'(\cdot)$ via the Bochner integral on $[a, b)$, i.e.*

$$\Omega(\lambda) = \Omega(a) + \int_a^\lambda \Omega'(x)dx, \quad \lambda \in [a, b). \tag{3.7}$$

Assume also that $F(\cdot)$ is a spectral measure in $\mathcal{H}$ with compact support, $\operatorname{supp}(F) \subseteq \Delta := [a, b)$, and $\varphi \in C[c_1, c_2]$. Assume also that commutation relation (3.2) holds. Then the operator-valued function $\varphi(\Omega(\cdot))$ is F-admissible and

$$\int_\Delta \varphi(\Omega(\lambda))F(d\lambda) = \varphi(\widehat{X}). \tag{3.8}$$

Proof. It is known (see[6] (Proposition 5.1.4)) that the integral (3.1) exists whenever $\Omega(\cdot)$ admits representation (3.7). By Proposition 3.3(iii) $\varphi(X(\cdot))$ is F-admissible and equality (3.8) holds. □

Remark 3.6 *Emphasize that absolute continuity of $\Omega(\cdot)$ (and even its Lipschitz property) does not ensure representation (3.7) (see[32] (Chapter 5)). Thus, the conditions in both corollaries are different.*

If $F(\cdot)$ is a spectral measure on $\mathbb{R}$ with non-compact support, then we define improper operator spectral integrals by

$$\int_{\mathbb{R}} \Omega(\lambda)F(d\lambda) := \text{s-}\lim_{\substack{b\to+\infty \\ a\to-\infty}} \int_\Delta \Omega(\lambda)F(d\lambda),$$

$$\int_{\mathbb{R}} F(d\lambda)\Omega(\lambda) := \text{s-}\lim_{\substack{b\to+\infty \\ a\to-\infty}} \int_\Delta F(d\lambda)\Omega(\lambda).$$

Obviously, the improper operator spectral integral $\int_{\mathbb{R}} \Omega(\lambda)F(d\lambda)$ exists if and only if the following conditions

$$\text{s-}\lim_{b\to\infty} \int_b^{b+\varepsilon} \Omega(\lambda)F(d\lambda) = 0 \quad \text{and} \quad \text{s-}\lim_{a\to-\infty} \int_{a-\varepsilon}^a \Omega(\lambda)F(d\lambda) = 0, \tag{3.9}$$

are satisfied for any $\varepsilon > 0$. Similar results hold true for $\int_{\mathbb{R}} F(d\lambda)\Omega(\lambda)$.

Proposition 3.7 *Let $\Omega : \mathbb{R} \longrightarrow [\mathcal{H}]$. Assume that $\Omega \restriction \Delta$ is F-admissible for every compact interval Δ and*

$$\| \Omega(\lambda) \| \le C_0(1 + |\lambda|)^\alpha, \qquad \lambda \in \mathbb{R},$$

for some constants $\alpha \ge 0$, $C_0 > 0$. Then the improper spectral integral $\int_{\mathbb{R}} \Omega(\lambda)F(d\lambda)f$ exists for any $f \in \mathcal{H}$ satisfying

$$\int_{\mathbb{R}} |\lambda|^{2\alpha} d \| F(\lambda)f \|^2 < \infty. \tag{3.10}$$

Proof. Let $b, c > 0$. Let $n \in \mathbb{N}$. Put $x_m := b + \frac{m-1}{n}c$, $\Delta_m := [x_m, x_m + \frac{c}{n})$, $\mathcal{Z} := \bigcup_{m=1}^n \Delta_m$. Then

$$\begin{aligned} \| \Sigma_{\mathcal{Z}}\Omega f \|^2 &= \Sigma_{m=1}^n \| \Omega(x_m)F(\Delta_m)f \|^2 \\ &\le C_0^2 \Sigma_{m=1}^n (1 + x_m)^{2\alpha} \| F(\Delta_m)f \|^2 \\ &\le C_0^2 \int_{[b,b+c)} (1 + \lambda)^{2\alpha} d \| F(\lambda)f \|^2 . \end{aligned}$$

Passing to the limit, as n tends to infinity, we get that

$$\| \int_{[b,b+c)} \Omega(\lambda)F(d\lambda)f \|^2 \le C_0^2 \int_{[b,b+c)} (1 + \lambda)^{2\alpha} d \| F(\lambda)f \|^2 .$$

The integral on the right hand side tends to zero, as b tends to infinity, provided (3.10) holds. The case $a \longrightarrow -\infty$ is treated similarly. □

4. BOUNDARY TRIPLETS FOR TENSOR PRODUCTS

A. Bounded case

Let A be a densely defined symmetric operator with equal deficiency indices acting in the separable Hilbert space $\mathfrak{H}$ and let T be a bounded self-adjoint operator acting on the separable Hilbert space $\mathfrak{T}$. Let us consider the closed symmetric operator $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$ in $\mathfrak{H}_S := \mathfrak{H} \otimes \mathfrak{T}$. We recall that the operator S is defined as the closure of $S_\odot := A \odot I_{\mathfrak{T}} + I_{\mathfrak{H}} \odot T$,

$$\mathrm{dom}\,(A \odot I_{\mathfrak{T}} + I_{\mathfrak{H}} \odot T) := \left\{ f = \sum_{k=1}^{n} g_k \otimes h_k : g_k \in \mathrm{dom}\,(A), h_k \in \mathfrak{T} \right\}$$

and

$$S_\odot f := \sum_{k=1}^{n} (Ag_k \otimes h_k + g_k \otimes Th_k), \qquad f \in \mathrm{dom}\,(A \odot I_{\mathfrak{T}} + I_{\mathfrak{H}} \odot T).$$

Obviously, the operator $S_\odot$ is densely defined and symmetric.

Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* with Gamma field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Let J_{A^*} be the embedding operator $J_{A^*} : \mathfrak{H}_+(A^*) \longrightarrow \mathrm{dom}\,(A^*)$. Obviously, $\mathrm{ran}\,(J_{A^*}) = \mathrm{dom}\,(A^*)$ and $\ker\,(J_{A^*}) = \{0\}$ as well as $\Gamma_j^A = \widehat{\Gamma}_j^A J_{A^*}^{-1}$, $j = 0, 1$. Notice that $\mathfrak{H}_+((A \otimes I_{\mathfrak{T}})^*) = \mathfrak{H}_+(A^* \otimes I_{\mathfrak{T}}) = \mathfrak{H}_+(A^*) \otimes \mathfrak{T}$ and $J_{(A \otimes I_{\mathfrak{T}})^*} = J_{A^* \otimes I_{\mathfrak{T}}}$. Moreover, one has

$$\mathrm{ran}\,(J_{(A \otimes I_{\mathfrak{T}})^*}) = \mathrm{dom}\,((A \otimes I_{\mathfrak{T}})^*) = \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}}).$$

We set

$$(\Gamma_j^A \,\widehat{\otimes}\, I_{\mathfrak{T}}) f := (\,\widehat{\Gamma}_j^A\, \otimes I_{\mathfrak{T}})\, J_{(A \otimes I_{\mathfrak{T}})^*}^{-1}, \quad j \in \{0, 1\}, \quad f \in \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}}).$$

It turns out that $\Pi_A \,\widehat{\otimes}\, I_{\mathfrak{T}} := \{\mathcal{H}^A \otimes \mathfrak{T}, \Gamma_0^A \,\widehat{\otimes}\, I_{\mathfrak{T}}, \Gamma_1^A \,\widehat{\otimes}\, I_{\mathfrak{T}}\}$ is a boundary triplet for $(A \otimes I_{\mathfrak{T}})^* = A^* \otimes I_{\mathfrak{T}}$.

Theorem 4.1 *Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* with γ-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Let also $T = T^* \in [\mathfrak{T}]$, and let Δ be the smallest closed interval containing the spectrum $\sigma(T)$. Finally, let $\widehat{E}_T(\delta) := I_{\mathcal{H}^A} \otimes E_T(\delta)$, $\delta \in \mathcal{B}(\mathbb{R})$, where $E_T(\cdot)$ is the spectral measure of T. Then:*

(i) *$\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\} := \Pi_A \,\widehat{\otimes}\, I_{\mathfrak{T}}$ is a boundary triplet for S^* such that $S_0 := S^* \upharpoonright \ker\,(\Gamma_0^S) = A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$.*

(ii) *The Gamma field $\gamma^S(\cdot)$ and the Weyl function $M^S(\cdot)$ of Π_S admit the following representations*

$$\gamma^S(z) = \int_\Delta \big(\gamma^A(z - \lambda) \otimes I_{\mathfrak{T}}\big)\, \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C}_\pm, \tag{4.1}$$

and

$$M^S(z) = \int_\Delta \widehat{E}_T(d\lambda)\, \big(M^A(z - \lambda) \otimes I_{\mathfrak{T}}\big) = \int_\Delta \big(M^A(z - \lambda) \otimes I_{\mathfrak{T}}\big)\, \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C}_\pm. \tag{4.2}$$

In particular,

$$\mathrm{ran}\, \left(\int_\Delta \big(\gamma^A(z - \lambda) \otimes I_{\mathfrak{T}}\big) \widehat{E}_T(d\lambda) \right) = \mathfrak{N}_z(S^*) = \ker\,(S^* - z). \tag{4.3}$$

(iii) *If the Weyl function $M^A(\cdot)$ is of scalar type, $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, then*

$$M^S(z) = I_{\mathcal{H}^A} \otimes m^A(z - T), \qquad z \in \mathbb{C}_\pm.$$

In particular, the latter holds whenever $n_\pm(A) = 1$.

Note that the integrals (4.2) and (4.1) exist due to Corollary 3.4 since both the Weyl function $M^S(z-\cdot)$ and γ-field $\gamma^S(z-\cdot)$ are holomorphic in λ, hence Lipschitz functions.

Proof. (i) The proof is straightforward.

(ii) In accordance with[2] (Lemma 7.2) both integrals (4.1) and (4.2) exist since the functions $\gamma^A(\cdot)$ and $M^A(\cdot)$ are Lipschitz. Let $\pi = \{a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = b\}$ be a partition of $\Delta = [a,b]$, $\Delta_k := [\lambda_{k-1}, \lambda_k)$, and let

$$T_k := \lambda_k E(\Delta_k), \qquad T_\pi := \bigoplus_{k=1}^n T_k = \sum_{k=1}^n \lambda_k E_T(\Delta_k), \qquad S_\pi := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T_\pi, \tag{4.4}$$

and $\mathfrak{T}_k := \operatorname{ran} E(\Delta_k)$. T_k is regarded as an operator in $\mathfrak{T}_k$. It is easily seen that $\mathfrak{T} = \oplus_1^n \mathfrak{T}_k$ and

$$S_\pi = \bigoplus_{k=1}^n S_k, \qquad S_k := A \otimes I_{\mathfrak{T}_k} + I_{\mathfrak{H}} \otimes T_k \in \mathcal{C}(\mathfrak{H} \otimes \mathfrak{T}_k). \tag{4.5}$$

Clearly, $S_k^* := A^* \otimes I_{\mathfrak{T}_k} + I_{\mathfrak{H}} \otimes T_k$. Moreover for every k such that $\mathfrak{T}_k \neq \{0\}$ we have $\sigma(T_k) = \{\lambda_k\}$ and hence $\sigma(S_k^*) = \sigma(A^* \otimes I_{\mathfrak{T}_k}) + \lambda_k$ and $\mathfrak{N}_z(S_k) = \mathfrak{N}_{z-\lambda_k}(A) \otimes \mathfrak{T}_k$.

Further, clearly,

$$S_\pi^* = A^* \otimes (\oplus_1^n E_T(\Delta_k)) + I_{\mathfrak{H}} \otimes (\oplus_1^n \lambda_k E_T(\Delta_k)) = \bigoplus_1^n (A^* + \lambda_k I_{\mathfrak{H}}) \otimes E_T(\Delta_k). \tag{4.6}$$

Hence $\mathfrak{N}_z(S_\pi) = \ker(S_\pi^* - zI_{\mathfrak{H}}) = \operatorname{ran}\left(\sum_{k=1}^n \gamma^A(z-\lambda_k) \otimes E_T(\Delta_k)\right)$

Noting that $\Gamma_0^{S_\pi} = \Gamma_0^S = \Gamma_0^A \otimes I_{\mathfrak{T}}$ and using definition (2.3) one gets

$$\Gamma_0^{S_\pi}\left(\sum_{k=1}^n \gamma^A(z-\lambda) \otimes E_T(\Delta_k)\right) = \sum_{k=1}^n \Gamma_0^A \gamma^A(z-\lambda_k) \otimes E_T(\Delta_k) = \sum_{k=1}^n I_{\mathcal{H}} \otimes E_T(\Delta_k) = I_{\mathcal{H}} \otimes I_{\mathfrak{T}} = I_{\mathcal{H}\otimes\mathfrak{T}}. \tag{4.7}$$

Combining this relation with definition (2.3) of the γ-field one derives

$$\gamma^{S_\pi}(z) = \left(\Gamma_0^{S_\pi} \upharpoonright \mathfrak{N}_z(S_\pi)\right)^{-1} = \sum_{k=1}^n \gamma^A(z-\lambda_k) \otimes E_T(\Delta_k). \tag{4.8}$$

Applying operator Γ_1 to this equality and using Definition 2.4 we arrive at the Weyl function $M^{S_\pi}(\cdot)$ corresponding to the triplet Π^{S_π} of the operator S_π^*,

$$M^{S_\pi}(z) = \Gamma_1^{S_\pi}\gamma^{S_\pi}(z) = \sum_{k=1}^n \Gamma_1^A \gamma^A(z-\lambda_k) \otimes E_T(\Delta_k) = \sum_{k=1}^n M^A(z-\lambda_k) \otimes E_T(\Delta_k). \tag{4.9}$$

Since the integrals (5.1) and (5.2) exist, the following uniform convergence holds

$$\gamma^{S_\pi}(z) \to \int_\Delta \left(\gamma^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda) =: \widetilde{\gamma}^S(z) \qquad \text{as} \quad |\pi| \to 0, \tag{4.10}$$

and

$$M^{S_\pi}(z) \to \int_\Delta \left(M^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda) =: \widetilde{M}^S(z) \qquad \text{as} \quad |\pi| \to 0, \tag{4.11}$$

where as usual $|\pi| = \max_{k=1,2,\ldots,n} |\Delta_k|$.

Next we show that $\widetilde{\gamma}^S(z) = \gamma^S(z)$ and $\widetilde{M}^S(z) = M^S(z)$ for $z \in \mathbb{C}_\pm$. One gets

$$\big((A^*-z) \otimes I_{\mathfrak{T}}\big)\gamma^{S_\pi}(z)g = \sum_{k=1}^n (A^*-z)\gamma^A(z-\lambda_k) \otimes E_T(\Delta_k)g$$

$$= -\sum_{k=1}^n \lambda_k \gamma^A(z-\lambda_k) \otimes E_T(\Delta_k)g \to -\int_\Delta \big(\lambda\gamma^A(z-\lambda) \otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda)g \qquad \text{as} \quad |\pi| \to 0, \tag{4.12}$$

as $|\pi| \to 0$. Since A^* is closed, one gets by combining this relation with (4.10) that $\int_\Delta \big(\lambda\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda)g \in \mathrm{dom}\,(A^*\otimes I_{\mathfrak{T}})$ for each $g \in \mathcal{H}\otimes\mathfrak{T}$ and

$$\big((A^*-z)\otimes I_{\mathfrak{T}}\big)\int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda) = -\int_\Delta \big(\lambda\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda). \tag{4.13}$$

In turn, using this relation and applying Proposition 3.2 we derive

$$\begin{aligned}(S^*-z)\int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda) &= \big((A^*-z)\otimes I\big)\int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda)\\ &\quad + \int_\Delta \lambda\widehat{E}_T(d\lambda)\cdot\int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda)\\ &= -\int_\Delta \big(\lambda\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda) + \int_\Delta \big(\lambda\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda) = 0. \end{aligned}\tag{4.14}$$

It follows that $\mathrm{ran}\,\Big(\int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\widehat{E}_T(d\lambda)\Big) \subset \mathfrak{N}_z(S^*) = \ker\,(S^*-z)$.

Let us show that the convergence in (4.10) holds in $\mathfrak{H}_+(S)$, i.e. in the graph norm.

Choose a sequence $\{\pi_n\}_1^\infty$ of partitions of $[a,b]$ such that $\lim_{n\to\infty}|\pi_n| = 0$. Since the convergence in (4.10) is uniform, there exists a constant $C(z) > 0$ depending on z and not depending on n and such that $\|\gamma^{S_{\pi_n}}(z)\| \le C(z)$ for all n. Besides, for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\|T_{\pi_n} - T\| \le \varepsilon$ for $n \ge N$. Taking these relations into account one gets

$$\begin{aligned}\|(S^*-z)\gamma^{S_\pi}(z)g\| &= \|(S^*-z)\gamma^{S_\pi}(z)g - (S^*_\pi - z)\gamma^{S_\pi}(z)g\|\\ &= \|\big(I\otimes(T-T_\pi)\big)\gamma^{S_\pi}(z)g\| \le \varepsilon\|\gamma^{S_\pi}(z)\|\cdot\|g\| \le \varepsilon C(z)\|g\|\end{aligned}\tag{4.15}$$

for any $\pi \in \{\pi_n\}_N^\infty$, hence $\|\lim_{n\to\infty}(S^*-z)\gamma^{S_{\pi_n}}(z)g\| = 0$ for any $g \in \mathcal{H}\otimes\mathfrak{T}$. In turn, combining this relation with (4.10) yields

$$\lim_{n\to\infty}\left\|\gamma^{S_{\pi_n}}(z) - \int_\Delta \big(\gamma^A(z-\lambda)\otimes I_{\mathfrak{T}}\big)\,\widehat{E}_T(d\lambda)\right\|_{\mathfrak{H}_+(S)} = 0. \tag{4.16}$$

It follows from (4.7) that $\Gamma_0^S\gamma^{S_\pi}(z) = \Gamma_0^{S_\pi}\gamma^{S_\pi}(z) = I_{\mathcal{H}}\otimes I_{\mathfrak{T}} \to I_{\mathcal{H}}\otimes I_{\mathfrak{T}}$ as $|\pi| \to 0$. Therefore relation (4.16) implies

$$\Gamma_0^S\widetilde{\gamma}^S(z) = I_{\mathcal{H}}\otimes I_{\mathfrak{T}},$$

i.e. $\widetilde{\gamma}^S(z) = \gamma^S(z)$. This proves (4.1). In turn, (4.1) implies (4.3).

Further, combining just established relation $\widetilde{\gamma}^S(\cdot) = \gamma^S(\cdot)$ with relation (4.16) and using the boundedness of the operator $\Gamma_1^S \in [\mathfrak{H}_+(S),\mathcal{H}]$ we obtain

$$\lim_{n\to\infty} M^{S_{\pi_n}}(z) = \lim_{n\to\infty}\Gamma_1^{S_{\pi_n}}\gamma^{S_{\pi_n}}(z) = \lim_{n\to\infty}\Gamma_1^S\gamma^{S_{\pi_n}}(z) = \Gamma_1^S\gamma^S(z) = M^S(z), \quad z \in \mathbb{C}_\pm, \tag{4.17}$$

where the convergence is uniform. In turn, combining this relation with (4.11) yields (4.2). □

Remark 4.2 *Another proof of Theorem 4.1 can be found in*[9] *(cf. Proposition 3.1 and 3.2).*

Example 4.3 Let us illustrate the theorem above. To this end we consider the case that A is a closed symmetric operator with deficiency indices $n_\pm = 2$. In particular, let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ where $\mathcal{H}^A = (\mathcal{H}_1^A \oplus \mathcal{H}_2^A)^t$, $\mathcal{H}_j^A = \mathbb{C}$, $j = 1,2$. We use the representation

$$\Gamma_j^A = \begin{pmatrix}\Gamma_{j1}^A\\ \Gamma_{j2}^A\end{pmatrix} : \mathrm{dom}\,(A^*) \longrightarrow \begin{matrix}\mathcal{H}_1^A\\ \oplus\\ \mathcal{H}_2^A\end{matrix}, \quad j = 0,1.$$

For the Gamma field $\gamma^A(\cdot)$ we use the representation $\gamma^A(z) = (\gamma_1^A(z), \gamma_2^A(z))$, $\gamma_j^A(z) : \mathcal{H}_j^A \longrightarrow \mathfrak{H}$, $j = 1,2$, $z \in \mathbb{C}_\pm$. The Weyl function $M^A(\cdot)$ admits the representation

$$M^A(z) = \begin{pmatrix} m_{11}^A(z) & m_{12}^A(z)\\ m_{21}^A(z) & m_{22}^A(z)\end{pmatrix}, \quad z \in \mathbb{C}_\pm, \tag{4.18}$$

where $m_{ij}^A(\cdot)$ are holomorphic functions in $\mathbb{C}_\pm$.

We consider the closed symmetric operator $S = A\otimes I_{\mathfrak{T}} + I_{\mathfrak{H}}\otimes T$, where T is bounded and self-adjoint. Let $\Pi_S = \Pi_A \widehat{\otimes} I_{\mathfrak{T}}$, cf. Theorem 4.1 (i). Obviously, the boundary value space $\mathcal{H}^S = \mathcal{H}^A\otimes\mathfrak{T}$ can be decomposed by $\mathcal{H}^S = (\mathcal{H}_1^S\oplus\mathcal{H}_2^S)^t$, $\mathcal{H}_j^S := \mathfrak{T}$, $j=1,2$. The boundary value maps $\Gamma_0^S = \Gamma_0^A\widehat{\otimes} I_{\mathfrak{T}}$ and $\Gamma_1^S = \Gamma_1^A\widehat{\otimes} I_{\mathfrak{T}}$ will be represented by

$$\Gamma_0^S = \begin{pmatrix}\Gamma_{01}^S\\ \Gamma_{02}^S\end{pmatrix} : \mathrm{dom}\,(S^*) \longrightarrow \begin{matrix}\mathcal{H}_1^S\\ \oplus\\ \mathcal{H}_2^S\end{matrix} \text{ and } \Gamma_1^S = \begin{pmatrix}\Gamma_{11}^S\\ \Gamma_{12}^S\end{pmatrix} : \mathrm{dom}\,(S^*) \longrightarrow \begin{matrix}\mathcal{H}_1^S\\ \oplus\\ \mathcal{H}_2^S\end{matrix} \tag{4.19}$$

where $\Gamma_{0j}^S := \Gamma_{0j}^A\widehat{\otimes} I_{\mathfrak{T}}$ and $\Gamma_{1j}^S := \Gamma_0^A\widehat{\otimes} I_{\mathfrak{T}}$, $j=0,1$. From (4.1) we get the representation $\gamma^S(z) = (\gamma_1^S(z),\gamma_2^S(z))$, $z\in\mathbb{C}_\pm$, where $\gamma_j^S(z) : \mathcal{H}_j^S \longrightarrow \mathfrak{H}$,

$$\gamma_j^S(z) := \int_\Delta \gamma_j^A(z-\lambda)\,\widehat{E}_T(d\lambda), \quad j=1,2.$$

The Weyl function $M^S(\cdot)$ admits the representation

$$M^S(z) = \begin{pmatrix} m_{11}^A(z-T) & m_{12}^A(z-T)\\ m_{21}^A(z-T) & m_{22}^A(z-T)\end{pmatrix} : \begin{matrix}\mathcal{H}_1^S\\ \oplus\\ \mathcal{H}_2^S\end{matrix} \longrightarrow \begin{matrix}\mathcal{H}_1^S\\ \oplus\\ \mathcal{H}_2^S\end{matrix}, \quad z\in\mathbb{C}_\pm. \tag{4.20}$$

The representation of the Weyl function becomes very simple if $M^A(\cdot)$ is diagonal. In this case we have $M^S(z) = \mathrm{diag}\,(m_{11}^A(z-T), m_{22}^A(z-T))$, $z\in\mathbb{C}_\pm$. ◁

Let us compute the normalized boundary triplet $\widetilde{\Pi}_S$ associated with Π_S in accordance with Lemma 2.6.

Proposition 4.4 *Let $\Pi_A = \{\mathcal{H}^A,\Gamma_0^A,\Gamma_1^A\}$ be a boundary triplet for A^* with the γ-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Let also $A_0 := A^*\restriction\ker(\Gamma_0^A)$, $T=T^*\in[\mathfrak{T}]$, and let Δ be the smallest closed interval containing the spectrum $\sigma(T)$, and let $\Pi_S = \Pi_A\widehat{\otimes} I_{\mathfrak{T}}$. Finally, let $\widehat{E}_T(\delta) := I_{\mathcal{H}^A}\otimes E_T(\delta)$, $\delta\in\mathcal{B}(\mathbb{R})$, where $E_T(\cdot)$ is the spectral measure of T. Then:*

(i) *The triplet $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S,\widetilde{\Gamma}_0^S,\widetilde{\Gamma}_1^S\}$ with $\widetilde{\mathcal{H}}^S := \mathcal{H}^A\otimes\mathfrak{T}$ and*

$$\begin{aligned}\widetilde{\Gamma}_0^S &:= \left(\int_\Delta \widehat{E}_T(d\lambda)\sqrt{\mathrm{Im}(M^A(i-\lambda))}\otimes I_{\mathfrak{T}}\right)\cdot(\Gamma_0^A\widehat{\otimes} I_{\mathfrak{T}}),\\ \widetilde{\Gamma}_1^S &:= \left(\int_\Delta \widehat{E}_T(d\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}}\right)\cdot\\ &\quad\cdot\left(\Gamma_1^A\widehat{\otimes} I_{\mathfrak{T}} - \left(\int_\Delta \widehat{E}_T(d\lambda)\mathrm{Re}(M^A(i-\lambda))\otimes I_{\mathfrak{T}}\right)\cdot(\Gamma_0^A\widehat{\otimes} I_{\mathfrak{T}})\right)\end{aligned} \tag{4.21}$$

forms a normalized boundary triplet for S^* such that

$$\widetilde{S}_0 := S^*\restriction\ker(\widetilde{\Gamma}_0^S) = S_0 = A_0\otimes I_{\mathfrak{T}} + I_{\mathfrak{H}}\otimes T. \tag{4.22}$$

(ii) *The γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ corresponding to the normalized boundary triplet $\widetilde{\Pi}_S$ admit the following representations*

$$\widetilde{\gamma}^S(z) = \int_\Delta\left(\gamma^A(z-\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}}\right)\,\widehat{E}_T(d\lambda), \quad z\in\mathbb{C}_\pm, \tag{4.23}$$

and

$$\widetilde{M}^S(z) = \int_\Delta (L^A(z-\lambda,i-\lambda)\otimes I_{\mathfrak{T}})\widehat{E}_T(d\lambda) = \int_\Delta \widehat{E}_T(d\lambda)(L^A(z-\lambda,i-\lambda)\otimes I_{\mathfrak{T}}), \qquad z\in\mathbb{C}_\pm, \tag{4.24}$$

where

$$L^A(z,\zeta) := \frac{1}{\sqrt{\mathrm{Im}(M^A(\zeta))}}(M^A(z)-\mathrm{Re}(M^A(\zeta)))\frac{1}{\sqrt{\mathrm{Im}(M^A(\zeta))}}, \qquad z\in\mathbb{C}_\pm, \zeta\in\mathbb{C}_+. \tag{4.25}$$

(iii) *If the Weyl function $M^A(\cdot)$ is of scalar type, $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, $m^A(\cdot) \in R[\mathbb{C}]$, then*

$$\widetilde{M}^S(z) = I_{\mathcal{H}^A} \otimes \frac{m^A(z-T) - \mathrm{Re}(m^A(i-T))}{\mathrm{Im}(m^A(i-T))}, \qquad z \in \mathbb{C}_\pm. \tag{4.26}$$

In particular, the latter happen whenever $n_\pm(A) = 1$.

Proof. (i) By Theorem 4.1, $M^S(z) = \int_\Delta M^A(z-\lambda) \otimes I_{\mathcal{T}} \widehat{E}_T(d\lambda)$, and hence for each $z \in \mathbb{C}_+$

$$\mathrm{Im}(M^S(z)) = \int_\Delta \big(\mathrm{Im}(M^A(z-\lambda)) \otimes I_{\mathcal{T}}\big) \widehat{E}_T(d\lambda) \quad \text{and} \quad \mathrm{Re}(M^S(z)) = \int_\Delta \big(\mathrm{Re}(M^A(z-\lambda)) \otimes I_{\mathcal{T}}\big) \widehat{E}_T(d\lambda). \tag{4.27}$$

First we note that both integrals in (4.27) exist since the operator-valued functions $\mathrm{Im}(M^A(z-\cdot))$ and $\mathrm{Re}(M^A(z-\cdot)$ are Lipschitz (see[2]). Moreover, since the spectral measure $\widehat{E}_T = I_{\mathfrak{H}} \otimes E_T$ commutes with $M^A(z-\lambda) \otimes I_{\mathcal{T}}$, both functions $\mathrm{Im}(M^A(i-\cdot)) \otimes I_{\mathcal{T}}$ and $\mathrm{Re}(M^A(i-\cdot)) \otimes I_{\mathcal{T}}$ are $\widehat{E}_T$-admissible. Noting that $M^A(\cdot)$ is holomorphic on $\mathbb{C}_+$ and $0 \in \rho(\mathrm{Im} M(z))$ for $z \in \mathbb{C}_+$, one easily concludes that the operator-valued functions $\mathrm{Im}(M^A(i-\cdot)) \otimes I_{\mathcal{T}}$, $\mathrm{Re}(M^A(i-\cdot)) \otimes I_{\mathcal{T}}$, and $(\mathrm{Im}(M^A(i-\cdot))^{-1} \otimes I_{\mathcal{T}}$ are bounded on the compact set Δ and with some constants $c_1, c_2 > 0$ the following estimates hold

$$0 < c_1 \le \mathrm{Im}(M^A(i-\lambda)) \otimes I_{\mathcal{T}} \le c_2 \quad \text{and} \quad c_2^{-1} \le (\mathrm{Im}(M^A(i-\lambda))^{-1} \otimes I_{\mathcal{T}} \le c_1^{-1}, \qquad \lambda \in \Delta.$$

Since the function $\varphi(\cdot) = \sqrt{\cdot}$ is continuous on $\mathbb{R}_+$, then in accordance with Proposition 3.3(iii) the compositions $(\mathrm{Im}(M^A(i-\lambda)))^{1/2} \otimes I_{\mathcal{T}}$ and $(\mathrm{Im}(M^A(i-\lambda)))^{-1/2} \otimes I_{\mathcal{T}}$ are $\widehat{E}_T$-admissible and

$$\begin{aligned} R &:= \sqrt{\mathrm{Im}(M^S(i))} = \int_\Delta \left(\sqrt{\mathrm{Im}(M^A(i-\lambda))} \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda), \\ R^{-1} &= \frac{1}{\sqrt{\mathrm{Im}(M^S(i))}} = \int_\Delta \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda). \end{aligned} \tag{4.28}$$

Combining the second formula in (4.28) with formula (4.2) and applying Proposition 3.2 one arrives at

$$R^{-1}Q := R^{-1}\mathrm{Re}(M^S(i)) = \int_\Delta \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \mathrm{Re}(M^A(i-\lambda)) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda). \tag{4.29}$$

Now it follows from Lemma 2.6 (see formula (2.14)) that a triplet $\widetilde{\Pi}_S = \{\mathcal{H}_S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$, where

$$\widetilde{\Gamma}_0^S = \sqrt{\mathrm{Im}(M^S(i))}\Gamma_0^S \quad \text{and} \quad \widetilde{\Gamma}_1^S = \frac{1}{\sqrt{\mathrm{Im}(M^S(i))}}(\Gamma_1^S - \mathrm{Re}(M^S(i))\Gamma_0^S),$$

is a (normalized) boundary triplet for S^*. Combining these formulas with formulas (4.28) yields (4.21).

(ii) Combining (4.1) with the second identity in (4.28) and applying Proposition 3.2 we arrive at

$$\begin{aligned} \widetilde{\gamma}^S(z) = \gamma^S(z)R^{-1} &= \int_\Delta \big(\gamma^A(z-\lambda) \otimes I_{\mathfrak{T}}\big) \widehat{E}_T(d\lambda) \cdot \int_\Delta \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\mu))}} \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\mu) \\ &= \int_\Delta \left(\gamma^A(z-\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C}_\pm, \end{aligned}$$

which proves (4.23).

Similarly, combining formula (4.2) with the third formula in (4.28) and applying Proposition 3.2 implies

$$\begin{aligned} &\frac{1}{\sqrt{\mathrm{Im}(M^S(i))}} \big(M^S(z) - \mathrm{Re}(M^S(i))\big) \frac{1}{\sqrt{\mathrm{Im}(M^S(i))}} \\ &\quad = \int_\Delta \frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \big(M^A(z-\lambda) - \mathrm{Re}(M^A(i-\lambda)\big) \frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \widehat{E}_T(d\lambda), \quad z \in \mathbb{C}_\pm. \end{aligned}$$

This proves (4.24). Moreover, inserting in (4.24) $z = i$ one easily gets the equality $\widetilde{M}^S(i) = i(I_{\mathcal{H}^A} \otimes I_{\mathfrak{T}}) = iI_{\mathcal{H}^S}$ meaning that the triplet $\widetilde{\Pi}_S$ is normalized.

(iii) Representation (4.26) is immediate from (4.24). □

B. Unbounded case

Let A be a closed densely defined symmetric operator with equal deficiency indices in $\mathfrak{H}$ and let T be an unbounded self-adjoint operator in $\mathfrak{T}$. First we introduce an operator $S' := A \odot I_{\mathfrak{T}} + I_{\mathfrak{H}} \odot T$ by setting (cf.[31] (Chapter 7.5.2))

$$S'f := A \odot I_{\mathfrak{T}} f + I_{\mathfrak{H}} \odot Tf := \sum_{k=1}^{l} (Ag_k \otimes h_k) + \sum_{k=1}^{l} (g_k \otimes Th_k), \quad f = \sum_{k=1}^{l} g_k \otimes h_k \in \mathrm{dom}\,(S'),$$
$$\mathrm{dom}\,(S') := \left\{ f = \sum_{k=1}^{l} g_k \otimes h_k : g_k \in \mathrm{dom}\,(A), \quad h_k \in \mathrm{dom}\,(T), \quad l \in \mathbb{N} \right\}.$$

Clearly, S' is a densely defined symmetric operator. Further, we define the operator $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_A} \otimes T$ on $\mathfrak{K} := \mathfrak{H} \otimes \mathfrak{T}$ as the closure of S', i.e.

$$S := \overline{S'} := \overline{A \odot I_{\mathfrak{T}} + I_{\mathfrak{H}} \odot T}.$$

Denote by $\mathfrak{H}_+(A)$ the Hilbert space obtained by equipping the domain $\mathrm{dom}\,(A)$ with the graph norm. Let $J_A : \mathfrak{H}_+(A) \longrightarrow \mathfrak{H}$ be the embedding operator. Then $\mathrm{dom}\,(A \otimes I_T) = (J_A \otimes I_{\mathfrak{T}})(\mathfrak{H}_+(A) \otimes \mathfrak{T})$. By[31] (Proposition 7.26), $(A \otimes I_T)^* = A^* \otimes I_T$ and

$$\mathrm{dom}\,(A^* \otimes I_T) = (J_{A^*} \otimes I_{\mathfrak{T}})(\mathfrak{H}_+(A^*) \otimes \mathfrak{T}).$$

The operator $I_{\mathfrak{H}} \otimes T = \overline{I_{\mathfrak{H}} \odot T}$ is unbounded and self-adjoint. Moreover. one has

$$S = \overline{A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T} \qquad \text{and} \qquad \mathrm{dom}\,(S) \supseteq \mathcal{D} := \mathrm{dom}\,(A \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{H}} \otimes T).$$

Clearly, $\mathcal{D}$ is a core for S, i.e. $S = \overline{S \upharpoonright \mathcal{D}}$.

Further, setting $T_n := E_T((n, n{+}1])T$ and $\mathfrak{T}_n := E_T((n, n{+}1])\mathfrak{T}$, $n \in \mathbb{Z}$, one arrives at the orthogonal decomposition

$$T = \oplus_{n \in \mathbb{Z}} T_n, \quad \mathfrak{T} := \oplus_{n \in \mathbb{Z}} \mathfrak{T}_n,$$

where $T_n = T_n^* \in [\mathfrak{T}_n]$. Let $\mathfrak{K}_n := \mathfrak{H} \otimes \mathfrak{T}_n$, $n \in \mathbb{Z}$. Clearly, $\mathfrak{K} := \mathfrak{H} \otimes \mathfrak{T} = \oplus_{n \in \mathbb{Z}} \mathfrak{K}_n$. We set $S_n := A \otimes I_{\mathfrak{T}_n} + I_{\mathfrak{H}} \otimes T_n$, $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ the operator S_n is a well-defined closed symmetric operator in $\mathfrak{H}_n$.

Lemma 4.5 *Let A and T be as above. Let $T = \oplus_{n \in \mathbb{Z}} T_n$ be an orthogonal decomposition of T where $T_n = T_n^* \in [\mathfrak{T}_n]$. Then*

$$S = \oplus_{n \in \mathbb{Z}} S_n, \qquad S_n := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_n} \otimes T_n. \tag{4.30}$$

In particular, if T has a pure point spectrum, then $S = \oplus_{n \in \mathbb{Z}} S_n$ where $S_n = A \otimes I_{\mathfrak{T}_n} + \lambda_n I_{\mathfrak{H}_n}$, $\{\lambda_n\}_{n \in \mathbb{Z}}$ is the sequence of eigenvalues of T, and $\mathfrak{H}_n := \mathfrak{H} \otimes \mathfrak{T}_n$ with $\mathfrak{T}_n = E_T(\{\lambda_n\})\mathfrak{T}$.

Proof. The proof is obvious. □

In general, for any self-adjoint extension S_0 of S there is a boundary triplet $\Pi_S = \{\mathcal{H}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ such that $S_0 = S^* \upharpoonright \ker(\Gamma_0^S)$. Moreover, in accordance with Lemma 2.6 it is always possible starting with a Π_S to define a normalized boundary triplet $\widetilde{\Pi}_S$. In particular, we can find a boundary triplet Π_S for S^*, $S = A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$, such that $S_0 := A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. However, in applications we need a special boundary triplet feeling a tensor structure of the operators S and S^* and leading to simple forms of the corresponding Weyl function and γ-field.

Therefore in what follows we choose another strategy. Let Π_A be a boundary triplet for A^* with the corresponding γ-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Starting with this boundary triplet for A^* we construct a normalized boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ for S^* such that $S_0 = S^* \upharpoonright \ker(\Gamma_0^S)$ and the corresponding γ-field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ can be explicitly computed by means of $\gamma^A(\cdot)$ and $M^A(\cdot)$ (cf. the proof of Theorem 4.8).

Lemma 4.6 *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$. Let also $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* and let $M^A(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Further, let T be a self-adjoint operator on $\mathfrak{T}$ with spectral measure $E_T(\cdot)$ and let $\widehat{E}_T(\cdot) := I_{\mathcal{H}^A} \otimes E_T(\cdot)$. Then the following improper* *spectral integrals*

$$
\begin{aligned}
G_0 f &:= \int_{\mathbb{R}} \widehat{E}_T(d\lambda)\left(\sqrt{\mathrm{Im}(M^A(i-\lambda))}\otimes I_{\mathfrak{T}}\right) f \\
&= \int_{\mathbb{R}} \left(\sqrt{\mathrm{Im}(M^A(i-\lambda))}\otimes I_{\mathfrak{T}}\right)\widehat{E}_T(d\lambda) f
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
G_1 f &:= \int_{\mathbb{R}} \widehat{E}_T(d\lambda)\left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}}\right) f \\
&= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}}\right)\widehat{E}_T(d\lambda) f,
\end{aligned} \tag{4.32}
$$

$$
\begin{aligned}
G_2 f &:= \int_{\mathbb{R}} \widehat{E}_T(d\lambda)\left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\mathrm{Re}(M^A(i-\lambda))\otimes I_{\mathfrak{T}}\right) f \\
&= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\mathrm{Re}(M^A(i-\lambda))\otimes I_{\mathfrak{T}}\right)\widehat{E}_T(d\lambda) f
\end{aligned} \tag{4.33}
$$

exist for each $f \in \mathrm{dom}\,(I_{\mathcal{H}^A}\otimes T)$. *Moreover, the following improper spectral integrals*

$$
G(z)f := \int_{\mathbb{R}} \left(\gamma^A(z-\lambda)\ \frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}}\right)\widehat{E}_T(d\lambda) f, \tag{4.34}
$$

and

$$
\begin{aligned}
M(z)f &:= \int_{\mathbb{R}} \left(L^A(z-\lambda, i-\lambda)\otimes I_{\mathfrak{T}}\right)\widehat{E}_T(d\lambda) f \\
&= \int_{\mathbb{R}} \widehat{E}_T(d\lambda)\left(L^A(z-\lambda, i-\lambda)\otimes I_{\mathfrak{T}}\right) f, \qquad z\in\mathbb{C}_\pm
\end{aligned} \tag{4.35}
$$

exist for every $f \in \mathcal{H}^A\otimes\mathfrak{T}$, *where* $L^A(z,\zeta)$, $z\in\mathbb{C}_\pm$, $\zeta\in\mathbb{C}_+$, *is given by* (4.25).

Proof. We divide the proof in several steps. (i) Let $f \in \mathrm{dom}\,(I_{\mathcal{H}^A}\otimes T)$. Then

$$
\int_{\mathbb{R}} \lambda^2 d \parallel \widehat{E}_T(\lambda) f \parallel^2 < \infty.
$$

Note that in accordance with (2.8),

$$
\parallel (\mathrm{Im}(M^A(i-\lambda)))^{1/2}\otimes I_{\mathcal{T}} \parallel = O(|\lambda|) \quad \text{and} \quad \parallel (\mathrm{Im}(M^A(i-\lambda)))^{-1/2}\otimes I_{\mathcal{T}} \parallel = O(|\lambda|) \quad \text{as} \quad \lambda\to\infty.
$$

Therefore the convergence of the integrals in (4.31) and (4.32) is immediate from Proposition 3.7 with $\alpha = 1$.

(ii) To prove (4.33) it suffices to show that

$$
\parallel (\mathrm{Im}(M^A(i-\lambda)))^{-1/2}\mathrm{Re}(M^A(i-\lambda)) \parallel = O(|\lambda|) \quad \text{as} \quad \lambda\to\infty. \tag{4.36}
$$

Noting that

$$
(\mathrm{Im}(M^A(i-\lambda))^{-1/2}M^A(i-\lambda) = (\mathrm{Im}(M^A(i-\lambda))^{-1/2}\mathrm{Re}(M^A(i-\lambda)) + i(\mathrm{Im}(M^A(i-\lambda))^{1/2} \tag{4.37}
$$

and taking estimate (2.8) into account one concludes that the required estimate (4.36) is equivalent to the following one

$$
\parallel (\mathrm{Im}(M^A(i-\lambda))^{-1/2}M^A(i-\lambda) \parallel = O(|\lambda|) \qquad \text{as} \qquad \lambda\to\infty. \tag{4.38}
$$

Further, in accordance with (2.5)

$$
\mathrm{Im}(M^A(i-\lambda)) = -\mathrm{Im}(M^A(-i-\lambda)) = \gamma^A(-i-\lambda)^*\gamma^A(-i-\lambda) = \gamma^A(i-\lambda)^*\gamma^A(i-\lambda), \qquad \lambda\in\mathbb{R}.
$$

Hence there exists a family of isometries $V(\lambda \pm i)$ mapping $\mathcal{H}^A$ onto $\mathcal{N}_A(\pm i - \lambda) = \ker(A^* + \lambda \mp i)$ and such that

$$V(\lambda \pm i)(\mathrm{Im}(M^A(i-\lambda))^{1/2} = \gamma^A(\pm i - \lambda), \qquad \lambda \in \mathbb{R}. \tag{4.39}$$

Using (2.5), we get

$$M^A(i-\lambda) - M^A(i)^* = (2i-\lambda)\gamma^A(-i-\lambda)^*\gamma^A(-i) = (2i-\lambda)(\mathrm{Im}(M^A(i-\lambda))^{1/2}V(\lambda)^*\gamma^A(-i). \tag{4.40}$$

Thus

$$(\mathrm{Im}(M^A(i-\lambda))^{-1/2}M^A(i-\lambda) = (2i-\lambda)V(\lambda)^*\gamma^A(-i) + (\mathrm{Im}(M^A(i-\lambda))^{-1/2}M^A(-i). \tag{4.41}$$

Combining this relation with estimate (2.8) yields (4.38) as well as $\| (\mathrm{Im}(M^A(i-\lambda)))^{-1/2}\mathrm{Re}(M^A(i-\lambda)) \| = O(|\lambda|)$. To prove (4.33) it remains to apply Proposition 3.7 with $\alpha = 1$.

(iii) To prove the convergence of the integral (4.34) it suffices to show that

$$\| \gamma^A(z-\lambda)(\mathrm{Im}(M^A(i-\lambda))^{-1/2} \| \le \varkappa(z), \qquad \lambda \in \mathbb{R}. \tag{4.42}$$

with some positive constant $\varkappa(z) > 0$. In accordance with (2.4)

$$\gamma^A(z-\lambda) = (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1}\gamma^A(i-\lambda), \qquad z \in \mathbb{C}_\pm, \quad \lambda \in \mathbb{R}.$$

Moreover, it follows from (4.39) that

$$(\mathrm{Im}(M^A(i-\lambda))^{-1/2} = (\gamma^A(i-\lambda))^{-1}V(i+\lambda), \qquad \lambda \in \mathbb{R}. \tag{4.43}$$

Combining these relations yields

$$\gamma^A(z-\lambda)(\mathrm{Im}(M^A(i-\lambda))^{-1/2} = (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1}V(i+\lambda), \qquad z \in \mathbb{C}_\pm, \quad \lambda \in \mathbb{R}. \tag{4.44}$$

On the other hand

$$\| (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1} \| = \| I + (z-i)(A_0 + \lambda - z)^{-1} \| \le 1 + \frac{|z-i|}{|\mathrm{Im} z|} =: \varkappa(z).$$

Combining this estimate with identity (4.44) we arrive at the estimate (4.42). Proposition 3.7 with $\alpha = 1$ completes the proof.

(iv) To prove the existence of the integral (4.35) it suffices to show that for each fixed $z \in \mathbb{C}_\pm$

$$\| L^A(z-\lambda, i-\lambda) \| = O(1) \qquad \text{as} \qquad \lambda \to \infty, \tag{4.45}$$

and apply Proposition 3.7. It follows from (4.25) and identity (2.5) that

$$\begin{aligned} L^A(z-\lambda, i-\lambda) &= \frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}(M^A(z-\lambda) - M^A(i-\lambda))\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} + iI_{\mathcal{H}} \\ &= (z-i)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\gamma^A(-i-\lambda)^*\gamma^A(z-\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} + iI_{\mathcal{H}}, \quad z \in \mathbb{C}_\pm, \lambda \in \mathbb{R}. \end{aligned}$$

Inserting in this identity instead of $\gamma^A(-i-\lambda)^*$ its expression from (4.39) one gets

$$L^A(z-\lambda, i-\lambda) = (z-i)V(\lambda - i)^*\gamma^A(z-\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} + iI_{\mathcal{H}}. \tag{4.46}$$

Finally, combining this identity with (4.42) implies (4.45). □

Remark 4.7 *Combining estimates* (4.36) *and* (2.8) *we obtain*

$$\| \mathrm{Re}(M^A(i-\lambda)) \| \le \| (\mathrm{Im}(M^A(i-\lambda)))^{1/2} \| \cdot \| (\mathrm{Im}(M^A(i-\lambda)))^{-1/2}\mathrm{Re}(M^A(i-\lambda)) \| = O(|\lambda|^2). \tag{4.47}$$

as $\lambda \to \infty$. *Simple examples show that even for a scalar Nevanlinna function* $f \in R[\mathbb{C}]$ *the function* $\| (\mathrm{Im}(f(i-\lambda)))^{-1/2}\mathrm{Re}(f(i-\lambda)) \|$ *is not necessarily bounded.*

Theorem 4.8 *Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^*, let $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Let also $T = T^* \in \mathcal{C}(\mathfrak{T}) \setminus [\mathfrak{T}]$ and $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. Then:*

(i) *There exists a normalized boundary triplet $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ for S^* such that $\widetilde{\mathcal{H}}^S := \mathcal{H}^A \otimes \mathfrak{T}$ and $S_0 := S^* \upharpoonright \ker(\widetilde{\Gamma}_0^S) = A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$, and for any $f \in \mathfrak{D} := \mathrm{dom}\,(S^*) \cap \mathrm{dom}\,(I_{\mathfrak{H}} \otimes T)(\subseteq \mathrm{dom}\,(S^*))$*

$$
\begin{aligned}
\widetilde{\Gamma}_0^S f &:= \left(\int_{\mathbb{R}} \widehat{E}_T(d\lambda)\sqrt{\mathrm{Im}(M^A(i-\lambda))} \otimes I_{\mathfrak{T}}\right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}}) f,\\
\widetilde{\Gamma}_1^S f &:= \left(\int_{\mathbb{R}} \widehat{E}_T(d\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}}\right) \cdot (\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}}) f\\
&\quad - \left(\int_{\mathbb{R}} \widehat{E}_T(d\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\mathrm{Re}(M^A(i-\lambda)) \otimes I_{\mathfrak{T}}\right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}}) f.
\end{aligned} \tag{4.48}
$$

(ii) *The γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ corresponding to the triplet $\widetilde{\Pi}_S$ are given by*

$$
\widetilde{\gamma}^S(z) = G(z) \qquad \text{and} \qquad \widetilde{M}^S(z) = M(z), \quad z \in \mathbb{C}_\pm, \tag{4.49}
$$

where $G(\cdot)$ and $M(\cdot)$ are defined by (4.34) and (4.35), respectively.

(iii) *If $M^A(\cdot)$ is of scalar type, i.e. $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, then representation (4.26) remains true.*

Proof. (i) Clearly, $f \in \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}})$. Let $\Delta_n := [n, n+1)$, $n \in \mathbb{Z}$. We set $\mathfrak{T}_n := E_T(\Delta_n)\mathfrak{T}$ and $T_n = TE_T(\Delta_n)$, $n \in \mathbb{Z}$. Notice that $\mathfrak{T} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{T}_n$ and $T = \bigoplus_{n\in\mathbb{Z}} T_n$. Let also $R_{S_n} := \sqrt{\mathrm{Im}(M^{S_n}(i))}$ and $Q_{S_n} := \mathrm{Re}(M^{S_n}(i))$, $n \in \mathbb{Z}$. Then, by Proposition 4.4, a triplet $\widetilde{\Pi}_{S_n} = \{\mathcal{H}^{S_n}, \widetilde{\Gamma}_0^{S_n}, \widetilde{\Gamma}_1^{S_n}\}$ with

$$
\mathcal{H}^{S_n} := \mathcal{H}^A \otimes \mathfrak{T}_n, \quad \widetilde{\Gamma}_0^{S_n} = R_{S_n}(\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}_n}), \quad \text{and} \quad \widetilde{\Gamma}_1^{S_n} = R_{S_n}^{-1}\left(\Gamma_1^{S_n} - Q_{S_n}\Gamma_0^{S_n}\right) = R_{S_n}^{-1}\Gamma_1^{S_n} - R_{S_n}^{-1}Q_{S_n}\Gamma_0^{S_n}, \tag{4.50}
$$

is a boundary triplet for S_n^* for each $n \in \mathbb{Z}$. In turn, Theorem 2.7 ensures that the direct sum $\widetilde{\Pi}_S := \bigoplus_{n\in\mathbb{Z}} \widetilde{\Pi}_{S_n} = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ of boundary triplets is an ordinary (normalized) boundary triplet for $S^* = \oplus_{n\in\mathbb{Z}} S_n^*$.

Setting $R := \oplus_n R_{S_n}$, applying formula (4.28) and noting that, by Lemma 4.6, the improper spectral integral (4.31) exists one gets that for any $h = \oplus_n h_n \in \mathrm{dom}\,(I_{\mathcal{H}^A} \otimes T) = \oplus_n \mathrm{dom}\,(I_{\mathcal{H}^A} \otimes T_n)$

$$
\begin{aligned}
Rh &= \bigoplus_n R_{S_n} = \bigoplus_{n\in\mathbb{Z}} \sqrt{\mathrm{Im}(M^{S_n}(i))} h_n = \bigoplus_{n\in\mathbb{Z}} \int_{[n,n+1)} \widehat{E}_{T_n}(\lambda) \left(\sqrt{\mathrm{Im}(M^A(i-\lambda)} \otimes I_{\mathfrak{T}_n}\right) h_n\\
&= \text{s-}\lim_{\substack{p\to+\infty\\ q\to-\infty}} \int_{[q,p)} \widehat{E}_T(\lambda) \left(\sqrt{\mathrm{Im}(M^A(i-\lambda)} \otimes I_{\mathfrak{T}}\right) h = \int_{\mathbb{R}} \widehat{E}_T(\lambda) \left(\sqrt{\mathrm{Im}(M^A(i-\lambda)} \otimes I_{\mathfrak{T}}\right) = G_0 h.
\end{aligned} \tag{4.51}
$$

Note that applying formula (4.28) we have replaced the integral $\int_{[n,n+1]}$ by $\int_{[n,n+1)}$. The latter is possible since $n+1 \notin \sigma_p(T_n)$ for each $n \in \mathbb{Z}$.

Next, similarly to (4.51) and using the convergence of the improper spectral integral (4.32) one gets from (4.28)

$$
R^{-1}h = \bigoplus_n R_{S_n}^{-1} = \bigoplus_{n\in\mathbb{Z}} \left(\sqrt{\mathrm{Im}(M^{S_n}(i))}\right)^{-1} h_n = \int_{\mathbb{R}} \widehat{E}_T(d\lambda) \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}}\right) h = G_1 h. \tag{4.52}
$$

Further, setting $Q := \oplus_n Q_{S_n} := \oplus_n \mathrm{Re}(M^{S_n}(i))$, applying formula (4.29) with Δ_n in place of Δ, and noting that by Lemma 4.6 the improper spectral integral (4.33) exists, we derive

$$
\begin{aligned}
R^{-1}Qh &= \bigoplus_{n\in\mathbb{Z}} R_{S_n}^{-1}\mathrm{Re}(M^{S_n}(i))h_n = \bigoplus_{n\in\mathbb{Z}} \int_{[n,n+1)} \widehat{E}_{T_n}(\lambda) \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\mathrm{Re}(M^A(i-\lambda)) \otimes I_{\mathfrak{T}_n}\right) h_n\\
&= \int_{\mathbb{R}} \widehat{E}_T(\lambda) \left(\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\mathrm{Re}(M^A(i-\lambda)) \otimes I_{\mathfrak{T}}\right) h = G_2 h.
\end{aligned} \tag{4.53}
$$

Further, let $f = \{f_n\}_{n\in\mathbb{Z}} \in \mathfrak{D} \subseteq \mathrm{dom}\,(A^*\otimes I_{\mathfrak{T}})$, $f_n \in \mathfrak{H}_A\otimes\mathfrak{T}_n$, $n\in\mathbb{Z}$. Note that $f \in \mathrm{dom}\,(\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}})\cap\mathrm{dom}\,(\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}})$ because $f\in\mathrm{dom}\,(A^*\otimes I_{\mathfrak{T}})$. Hence

$$(\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}})f = \bigoplus_{n\in\mathbb{Z}}(\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}_n})f_n \quad\text{and}\qquad (\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}})f = \bigoplus_{n\in\mathbb{Z}}(\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}_n})f_n.$$

On the other hand, by Theorem 2.7 (see formula (2.16))

$$\widetilde{\Gamma}_0^S f = R(\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}})f \quad\text{and}\quad \widetilde{\Gamma}_1^S f = R^{-1}(\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}})f + R^{-1}Q(\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}})f, \qquad f\in\mathfrak{D}.$$

Inserting in these relations instead of R, R^{-1}, and $R^{-1}Q$ their expressions from (4.51), (4.52), and (4.53), one arrives at formulas (4.48).
(ii) In accordance with Proposition 4.4(ii) the γ-field and Weyl function corresponding to the triplet $\widetilde{\Pi}_{S_n} = \{\mathcal{H}^{S_n}, \widetilde{\Gamma}_0^{S_n}, \widetilde{\Gamma}_1^{S_n}\}$ are given by

$$\widetilde{\gamma}^{S_n}(z) = \int_{[n,n+1)}\left(\gamma^A(z-\lambda)\frac{1}{\sqrt{\mathrm{Im}(M^A(i-\lambda))}}\otimes I_{\mathfrak{T}_n}\right)\widehat{E}_{T_n}(d\lambda), \quad z\in\mathbb{C}_\pm,$$

and

$$\widetilde{M}^{S_n}(z) = \int_{[n,n+1)}\left(L^A(z-\lambda, i-\lambda)\otimes I_{\mathfrak{T}_n}\right)\widehat{E}_{T_n}(d\lambda) = \int_{[n,n+1)}\widehat{E}_{T_n}(d\lambda)\left(L^A(z-\lambda,i-\lambda)\otimes I_{\mathfrak{T}_n}\right), \quad z\in\mathbb{C}_\pm,$$

respectively. Here $L^A(z,\zeta)$ is given by (4.25). Further, applying Theorem 2.7 (see formula (2.18)) and taking into account formulas (4.34) and (4.35), we arrive at (4.49).
(iii) This statement is now immediate from formula (4.26) and representation $T = \bigoplus_{n\in\mathbb{Z}} T_n$ with $T_n \in [\mathfrak{T}_n]$. □

C. Remarks

Remark 4.9
(i) If T is pure point, $\sigma(T) = \sigma_{pp}(T) = \{\lambda_k\}_{k\in\mathbb{Z}}$, then the boundary space $\mathcal{H}^S$ admits the representation $\mathcal{H}^S = \oplus_{k\in\mathbb{Z}}\mathcal{H}_k$, where $\mathcal{H}_k = \mathcal{H}^A\otimes\mathfrak{T}_k$ and $\mathfrak{T}_k$ is the eigenspace which corresponds to λ_k. One easily checks that the Weyl function admits the representation

$$M^S(z) = \bigoplus_{k\in\mathbb{Z}}\left(L(z-\lambda_k, i-\lambda_k)\otimes I_{\mathfrak{T}_k}\right), \quad z\in\mathbb{C}_\pm.$$

(ii) The set $\mathfrak{D} := \mathrm{dom}\,(S^*)\cap\mathrm{dom}\,(I_{\mathfrak{H}_A}\otimes T) \subseteq \mathrm{dom}\,(S^*)$ is a core for S^*. Equivalently this means that $\mathfrak{D}$ regarded as a subset $\widehat{\mathfrak{D}}$ of $\mathfrak{H}_+(S^*)$ is dense in the Hilbert space $\mathfrak{H}_+(S^*)$. Let $J_{S^*} : \mathfrak{H}_+(S^*) \longrightarrow \mathfrak{H}_S = \mathfrak{H}_A\otimes\mathfrak{T}$ be the embedding operator. We set $\widehat{\Gamma}^S_j := \Gamma_j^S J_{S^*} : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}^S$, $j\in\{0,1\}$. The operator $\widehat{\Gamma}^S_j$, $j\in\{0,1\}$, is bounded. Hence

$$\widehat{\Gamma}^S_j = \overline{\Gamma_j^S J_{S^*}\upharpoonright\widehat{\mathfrak{D}}}, \quad j\in\{0,1\}.$$

In other words, the closure of the operator $\Gamma_j^S\upharpoonright\mathfrak{D}$, $j\in\{0,1\}$, with respect to the topology of $\mathfrak{H}_+(S^*)$ gives Γ_j^S, $j\in\{0,1\}$. ◁

Remark 4.10 The case of a scalar type Weyl function can be slightly extended. Let us assume that there is a boundary triplet $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ of A^* such that $\mathcal{H}^A = \bigoplus_{k=1}^{n(A)}\mathcal{H}_k^A$, $\mathcal{H}_k^A := \mathbb{C}$, $n(A) := n_\pm(A)$. With respect to this decomposition we suppose that the Weyl function $M^A(\cdot)$ is diagonal, that is, it admits the representation

$$M^A(z) = \mathrm{diag}\,(m_1(z), m_2(z), \ldots, m_{n(A)}(z))$$
$$= \begin{pmatrix} m_1^A(z) & 0 & \cdots & \cdots \\ 0 & m_2^A(z) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ . & . & \cdots & m_{n(A)}^A(z)\end{pmatrix} : \begin{matrix}\mathcal{H}_1^A\\ \oplus \\ \mathcal{H}_2^A \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H}_{n(A)}^A\end{matrix} \longrightarrow \begin{matrix}\mathcal{H}_1^A\\ \oplus \\ \mathcal{H}_2^A \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H}_{n(A)}^A\end{matrix}, \quad z\in\mathbb{C}_\pm,$$

where $m_k(\cdot)$, $k = 1, 2, \ldots, n(A)$, are scalar Nevanlinna functions. If the Weyl function of a boundary triplet has this structure, then it is called of quasi scalar type. We are going to compute the boundary triplet Π_S as well Gamma field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ for the quasi scalar type case. We set

$$\Gamma^A_{jk} := P^{\mathcal{H}^A}_{\mathcal{H}^A_k}\Gamma_j : \mathrm{dom}\,(A^*) \longrightarrow \mathcal{H}^A_k, \quad j = 0,1, \quad k = 1, 2, \ldots, n(A).$$

Obviously, we have

$$\Gamma^A_{1k} f_z = m_k(z)\Gamma^A_{0k} f_z, \qquad f_z \in \ker\,(A^* - z), \quad k = 1, 2, \ldots, n(A).$$

Let us introduce the operator $\Gamma^A_{jk} \widehat{\otimes} I_{\mathfrak{T}} : \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}}) \longrightarrow \mathcal{H}^S_k := \mathcal{H}^A_k \otimes \mathfrak{T} = \mathfrak{T}$, $j = 0, 1$, $k = 1, 2, \ldots, n(A)$. Notice that

$$\Gamma^A_j \widehat{\otimes} I_{\mathfrak{T}} = \begin{pmatrix} \Gamma^A_{j1} \widehat{\otimes} I_{\mathfrak{T}} \\ \Gamma^A_{j2} \widehat{\otimes} I_{\mathfrak{T}} \\ \vdots \\ \Gamma^A_{jn(A)} \widehat{\otimes} I_{\mathfrak{T}} \end{pmatrix} : \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}}) \longrightarrow \begin{matrix} \mathcal{H}^S_1 \\ \oplus \\ \mathcal{H}^S_2 \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H}^S_{n(A)} \end{matrix}.$$

Notice that $\mathcal{H}^S = \mathcal{H}^A \otimes \mathfrak{T} = \bigoplus_{k=1}^{n(A)} \mathcal{H}^S_k$. Setting $\Gamma^S_{jk} := P^{\mathcal{H}^S}_{\mathcal{H}^S_k}\Gamma^S_j$, $j \in \{0, 1\}$, $k \in \{1, 2, \ldots, n(A)\}$, we get $\Gamma^S_j = (\Gamma^S_{j1}, \Gamma^S_{j2}, \ldots, \Gamma^S_{jn(A)})^t$, $j \in \{0, 1\}$. Using (4.48) we get

$$\begin{aligned} \Gamma^S_{0k} f &= \sqrt{\mathrm{Im}(m_k(i - T))}(\Gamma^A_{0k} \widehat{\otimes} I_{\mathfrak{T}}) f \\ \Gamma^S_{1k} f &= \frac{1}{\sqrt{\mathrm{Im}(m_k(i - T))}} \left(\Gamma^A_{1k} \widehat{\otimes} I_{\mathfrak{T}} - \mathrm{Re}(m_k(i - T))(\Gamma^A_{0k} \widehat{\otimes} I_{\mathfrak{T}})\right) f, \end{aligned} \tag{4.54}$$

$f \in \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{H}_A} \otimes T)$, $k \in \{1, 2, \ldots, n(A)\}$.

To compute the γ-field we set

$$\gamma^A_k(z) := \gamma^A(z) \upharpoonright \mathcal{H}^A_k, \quad \gamma^A(z) = (\gamma^A_1(z), \gamma^A_2(z), \ldots, \gamma^A_{n(A)}(z)),$$

$z \in \mathbb{C}_\pm$, and

$$\gamma^S_k(\cdot) = \gamma^S(\cdot) \upharpoonright \mathcal{H}^S_k, \quad \gamma^S(z) = (\gamma^S_1(z), \gamma^S_2(z), \ldots, \gamma^S_{n(A)}(z)),$$

$z \in \mathbb{C}_\pm$, where $\mathcal{H}^S_k := \mathcal{H}^A_k \otimes \mathfrak{T} = \mathfrak{T}$, $k \in \{1, 2, \ldots, n(A)\}$. From (4.34) we find

$$\gamma^S_k(z) = \gamma^A_k(z - T)\frac{1}{\sqrt{\mathrm{Im}(m_k(i - T))}}, \quad z \in \mathbb{C}_\pm, \quad k \in \{1, 2, \ldots, n(A)\}. \tag{4.55}$$

Finally, the Weyl function takes the form

$$\begin{aligned} &M^S(z) = \\ &\mathrm{diag}\left(\frac{m^A_1(z - T) - \mathrm{Re}(m_1(i - T))}{\mathrm{Im}(m_1(i - T))}, \ldots, \frac{m^A_{n(A)}(z - T) - \mathrm{Re}(m_{n(A}(i - T))}{\mathrm{Im}(m_{n(A)}(i - T))}\right), \qquad z \in \mathbb{C}_\pm. \end{aligned} \tag{4.56}$$

$\triangleleft$

5. SUMS OF TENSOR PRODUCTS WITH NON-NEGATIVE SUMMANDS

A. Boundary triplets in the case of non-negative operators A and T

Here we complete previous results assuming the operators A and T to be non-negative. We denote by $\widehat{A}_{\mathrm{F}}$ and $\widehat{A}_{\mathrm{K}}$ the *Friedrich's* and *Krein's* extension of A, respectively.

Theorem 5.1 *Let A be a non-negative symmetric operator in $\mathfrak{H}$ and let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0^A) = \widehat{A}_F$. Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Let also $T = T^* \in [\mathfrak{T}]$, $T \geq 0$ and let $S = A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. Finally, let $\widehat{E}_T(\cdot) := I_{\mathcal{H}^A} \otimes E_T(\cdot)$, where $E_T(\cdot)$ is the spectral measure of T. Then:*

(i) $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\} := \Pi_A \,\widehat{\otimes}\, I_{\mathfrak{T}} := \{\mathcal{H}^A \otimes \mathfrak{T}, \Gamma_0^A \,\widehat{\otimes}\, I_{\mathfrak{T}}, \Gamma_1^A \,\widehat{\otimes}\, I_{\mathfrak{T}}\}$ *is a boundary triplet for S^* such that*

$$S_0 := S^* \upharpoonright \ker(\Gamma_0^S) = \widehat{S}_F = \widehat{A}_F \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T.$$

(ii) *The γ-field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ of Π_S admit the following representations*

$$\gamma^S(z) = \int_\Delta \left(\gamma^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C} \setminus \Delta, \tag{5.1}$$

and

$$M^S(z) = \int_\Delta \widehat{E}_T(d\lambda) \left(M^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) = \int_\Delta \left(M^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C} \setminus \Delta, \tag{5.2}$$

where Δ is the smallest closed interval containing the spectrum $\sigma(T)$.

(iii) *If the Weyl function $M^A(\cdot)$ is of scalar type, $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, then*

$$M^S(z) = I_{\mathcal{H}^A} \otimes m^A(z - T), \qquad z \in \mathbb{C}_\pm. \tag{5.3}$$

In particular, the latter holds whenever $n_\pm(A) = 1$.

Proof. (i) It is immediate from the definition that $S_0 = S^* \upharpoonright \ker(\Gamma_0^S) = A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. It remains to apply Proposition 5.8.

Statements (ii) and (iii) are immediate from Theorem 4.1. □

Lemma 5.2 *Let A be a densely defined closed non-negative symmetric operator in $\mathfrak{H}$ and let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* and let $A_0 \geq 0$. Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Further, let T be a non-negative self-adjoint operator on $\mathfrak{T}$, let $E_T(\cdot)$ be its spectral measure, and let $\widehat{E}_T(\cdot) := I_{\mathcal{H}^A} \otimes E_T(\cdot)$. Then the following improper spectral integrals*

$$G_0^+ f := \int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \left(\sqrt{((M^A)'(a-\lambda))} \otimes I_{\mathfrak{T}}\right) f, \qquad a < 0, \tag{5.4}$$

$$G_1^+ f := \int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \left(\frac{1}{\sqrt{((M^A)'(a-\lambda))}} \otimes I_{\mathfrak{T}}\right) f, \qquad a < 0, \tag{5.5}$$

$$G_2^+ f := \int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \left(\frac{1}{\sqrt{(M^A)'(a-\lambda)}} M^A(a-\lambda) \otimes I_{\mathfrak{T}}\right) f, \qquad a < 0. \tag{5.6}$$

exist for each $f \in \operatorname{dom}(I_{\mathcal{H}^A} \otimes T)$. Moreover, the following improper spectral integrals

$$G(z) f := \int_{\mathbb{R}_+} \left(\gamma^A(z-\lambda) \, \frac{1}{\sqrt{(M^A)'(a-\lambda)}} \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda) f, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \quad a < 0, \tag{5.7}$$

and

$$M(z) f := \int_{\mathbb{R}_+} \left(L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{T}}\right) \widehat{E}_T(d\lambda) f = \int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \left(L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{T}}\right) f, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{5.8}$$

converge for every $f \in \mathcal{H}^A \otimes \mathfrak{T}$, where

$$L^A(z,a) := \frac{1}{\sqrt{(M^A)'(a)}} \left(M^A(z) - M^A(a)\right) \frac{1}{\sqrt{(M^A)'(a)}}, \qquad z \in \rho(A_0), \; a \in \mathbb{R}_-. \tag{5.9}$$

Proof. (i) First we prove the convergence of integral in (5.7). It follows from (2.5) that $(M^A)'(z) = \gamma^A(\overline{z})^*\gamma^A(z)$. Hence

$$(M^A)'(a-\lambda) = \gamma^A(a-\lambda)^*\gamma^A(a-\lambda), \qquad \lambda \in \mathbb{R}_+, \quad a < 0. \tag{5.10}$$

This identity implies the existence of an isometry $V(a-\lambda)$ mapping $\mathcal{H}$ onto $\mathfrak{N}_{a-\lambda}(A)$ and such that

$$V(a-\lambda)\sqrt{(M^A)'(a-\lambda)} = \gamma^A(a-\lambda), \qquad \lambda \in \mathbb{R}_+, \tag{5.11}$$

Further, in accordance with (2.4)

$$\gamma^A(z-\lambda) = (A_0 - a + \lambda)(A_0 - z + \lambda)^{-1}\gamma^A(a-\lambda) = U(a-\lambda, z-\lambda)\gamma^A(a-\lambda), \tag{5.12}$$

where $U(a-\lambda, z-\lambda) := (A_0 - a+\lambda)(A_0 - z+\lambda)^{-1} \upharpoonright \mathfrak{N}_{a-\lambda}(A)$. It is easily checked that $U(a-\lambda, z-\lambda)$ isomorphically maps $\mathfrak{N}_{a-\lambda}(A)$ onto $\mathfrak{N}_{z-\lambda}(A)$. Combining relation (5.12) with (5.11) yields

$$\begin{aligned} &\|\gamma^A(z-\lambda)\big((M^A)'(a-\lambda)\big)^{-1/2}\| = \|U(a-\lambda, z-\lambda)V(a-\lambda)\| \\ &= \|I + (z-a)(A_0 - z+\lambda)^{-1}\| \le \begin{cases} 1 + |z-a|\cdot|\mathrm{Im} z|^{-1}, & z \in \mathbb{C}\setminus\mathbb{R}, \\ 1 + |x-a|\cdot|x|^{-1}, & x \in \mathbb{R}_-. \end{cases} \end{aligned} \tag{5.13}$$

Here we have taken into account that $|x| \le |x-\lambda| = |x|+\lambda$. The latter estimate implies boundedness of the integrand in (5.7) for each $z \in \mathbb{C}\setminus\mathbb{R}_+$. It remains to apply Proposition 3.7 with $\alpha = 0$.

(ii) Let us prove that

$$C(z,a) := \sup_{\lambda\in\mathbb{R}_+} \|L^A(z-\lambda, a-\lambda)\| < \infty \quad \text{for each} \quad z \in \mathbb{C}\setminus\mathbb{R}_+ \quad \text{and} \quad a < 0. \tag{5.14}$$

Combining identity (2.5) with (5.12) yields

$$\begin{aligned} M^A(z-\lambda) - M^A(a-\lambda) &= (z-a)\gamma^A(a-\lambda)^*\gamma^A(z-\lambda) \\ &= (z-a)\gamma^A(a-\lambda)^*U(a-\lambda, z-\lambda)\gamma^A(a-\lambda). \end{aligned}$$

In turn, inserting this identity in (5.9) and using (5.11) one derives

$$\begin{aligned} L^A(z-\lambda, a-\lambda) &= (z-a)\tfrac{1}{\sqrt{(M^A)'(a-\lambda)}}\gamma^A(a-\lambda)^*U(a-\lambda, z-\lambda)\gamma^A(a-\lambda)\tfrac{1}{\sqrt{(M^A)'(a-\lambda)}} \\ &= (z-a)V(a-\lambda)^*U(a-\lambda, z-\lambda)V(a-\lambda), \quad \lambda \in \mathbb{R}_+. \end{aligned} \tag{5.15}$$

Noting that $V(a-\lambda)$ is an isometry for each $\lambda \in \mathbb{R}_+$ and using estimate (5.13) one arrives at estimate (5.14). To prove convergence of the integral (5.8) for each $f \in \mathcal{H}^A \otimes \mathfrak{T}$, it remains to apply Proposition 3.7 with $\alpha = 0$.

(iii) Let us prove convergence of integrals (5.4) and (5.5). Since $A_0 \ge 0$, integral representation (2.6) implies

$$(M^A)'(a-\lambda) = \int_{\mathbb{R}_+} \frac{d\Sigma_A(t)}{(t-a+\lambda)^2}, \qquad \lambda \in \mathbb{R}_+, \quad a < 0. \tag{5.16}$$

Using this representation instead of (2.7) one proves the following analog of estimate (2.8)

$$C_1(1+|\lambda|^2)^{-1}\mathrm{Im}M(i) \le (M^A)'(a-\lambda) \le C_2(1+|\lambda|^2)\mathrm{Im}M(i), \qquad \lambda \in \mathbb{R}_+. \tag{5.17}$$

Combining this estimate with inequality $\int_{\mathbb{R}_+} \lambda^2 d \parallel \widehat{E}_T(\lambda)f \parallel^2 < \infty$ characterizing $f \in \mathrm{dom}\,(I_{\mathcal{H}^A} \otimes T)$, and applying Proposition 3.7 with $\alpha = 1$ yields convergence of both integrals (5.4) and (5.5).

(iv) Due to Proposition 3.7 (with $\alpha = 1$) to prove (5.6) it suffices to show that

$$\|\big((M^A)'(a-\lambda)\big)^{-1/2} M^A(a-\lambda)\| = O(|\lambda|) \quad \text{as} \quad \lambda \to \infty. \tag{5.18}$$

In accordance with (2.5)

$$M^A(a-\lambda) = M^A(a) - \lambda\gamma^A(a-\lambda)^*\gamma^A(a). \tag{5.19}$$

Combining this identity with (5.10) we derive

$$\|\big((M^A)'(a-\lambda)\big)^{-1/2} M^A(a-\lambda)\| \le \|M^A(a)\|\cdot\|\big((M^A)'(a-\lambda)\big)^{-1/2}\| + |\lambda|\cdot\|V^*(a-\lambda)\cdot\gamma^A(a)\|. \tag{5.20}$$

Noting that $V(a-\lambda)$ is an isometry and taking (5.17) into account we arrive at estimate (5.18). □

Theorem 5.3 *Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^*, $A_0 := A^* \upharpoonright \ker(\Gamma_0^A)$, let $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Let also $T = T^* \in \mathcal{C}(\mathfrak{T})$ be an unbounded self-adjoint operator in $\mathfrak{T}$ and $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. Then:*

(i) *There exists a boundary triplet $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ for S^* such that $\widetilde{\mathcal{H}}^S := \mathcal{H}^A \otimes \mathfrak{T}$ and $S_0 := S^* \upharpoonright \ker(\widetilde{\Gamma}_0^S) = A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. If $f \in \mathfrak{D} := \mathrm{dom}\,(S^*) \cap \mathrm{dom}\,(I_{\mathfrak{H}} \otimes T) \subseteq \mathrm{dom}\,(S^*)$, then $f \in \mathrm{dom}\,(S^*) \cap \mathrm{dom}\,(A^* \otimes I_{\mathfrak{T}})$ and*

$$\begin{aligned}
\widetilde{\Gamma}_0^S f &:= \left(\int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \sqrt{(M^A)'(a-\lambda)} \otimes I_{\mathfrak{T}} \right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}}) f, \\
\widetilde{\Gamma}_1^S f &:= \left(\int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \frac{1}{\sqrt{(M^A)'(a-\lambda)}} \otimes I_{\mathfrak{T}} \right) (\Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}}) f \\
&\quad - \left(\int_{\mathbb{R}_+} \widehat{E}_T(d\lambda) \left(\frac{1}{\sqrt{(M^A)'(a-\lambda)}} M^A(a-\lambda) \otimes I_{\mathfrak{T}} \right) \right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}}) f.
\end{aligned} \tag{5.21}$$

(ii) *The γ-field $\widetilde{\gamma}^S(\cdot)$ and Weyl function $\widetilde{M}^S(\cdot)$ corresponding to $\widetilde{\Pi}_S$ are given by*

$$\widetilde{\gamma}^S(z) = G(z) \qquad \text{and} \qquad \widetilde{M}^S(z) = M(z), \qquad z \in \rho(S_0), \tag{5.22}$$

where $G(\cdot)$ and $M(\cdot)$ are defined by (5.7) and (5.8), respectively.

(iii) *If $M^A(\cdot)$ is a scalar type function, i.e. $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, then representation (4.26) remains true.*

Proof. (i) First we let $\Delta_n := [n-1, n)$, $\mathfrak{T}_n := E_T(\Delta_n)\mathfrak{T}$, and $T_n = T E_T(\Delta_n)$, $n \in \mathbb{N}$. Clearly, $\mathfrak{T} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{T}_n$ and $T = \bigoplus_{n \in \mathbb{Z}} T_n$. We also put $S_n := A \otimes I_{\mathfrak{T}_n} + I_{\mathfrak{H}} \otimes T_n \in \mathcal{C}(\mathfrak{H} \otimes \mathfrak{T}_n)$. Clearly, each T_n is bounded and $\sigma(T_n) \subset [n-1, n]$.

By Theorem 5.1, $\Pi_{S_n} = \{\mathcal{H}^{S_n}, \Gamma_0^{S_n}, \Gamma_1^{S_n}\} := \Pi_A \widehat{\otimes} I_{\mathfrak{T}_n} := \{\mathcal{H}^A \otimes \mathfrak{T}_n, \Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}_n}, \Gamma_1^A \widehat{\otimes} I_{\mathfrak{T}_n}\}$ is a boundary triplet for S_n^* such that

$$S_{0n} := S^* \upharpoonright \ker(\Gamma_0^{S_n}) = A_0 \otimes I_{\mathfrak{T}_n} + I_{\mathfrak{H}} \otimes T_n, \qquad n \in \mathbb{N}.$$

Let also $M^{S_n}(\cdot)$ be the corresponding Weyl function. It follows from (5.2) that

$$(M^{S_n})'(z) = \int_{\Delta_n} \left((M^A)'(z-\lambda) \otimes I_{\mathfrak{T}_n} \right) \widehat{E}_T(d\lambda), \qquad z \in \mathbb{C} \setminus \Delta_n. \tag{5.23}$$

Since the function $\varphi(\cdot) = \sqrt{\cdot}$ is continuous on $\mathbb{R}_+$, then in accordance with Proposition 3.3(iii) the compositions $((M^A)'(a-\lambda))^{1/2} \otimes I_{\mathfrak{T}_n}$ and $((M^A)'(a-\lambda))^{-1/2} \otimes I_{\mathfrak{T}_n}$ are $\widehat{E}_{T_n}$-admissible. Therefore combining representation (5.23) with Proposition 3.3(iii) yields

$$\begin{aligned}
R_n := \sqrt{(M^{S_n})'(a)} &= \int_{\Delta_n} \left(\sqrt{(M^A)'(a-\lambda)} \otimes I_{\mathfrak{T}_n} \right) \widehat{E}_T(d\lambda), \qquad a < 0, \\
R_n^{-1} = \frac{1}{\sqrt{(M^{S_n})'(a)}} &= \int_{\Delta_n} \left(\frac{1}{\sqrt{(M^A)'(a-\lambda)}} \otimes I_{\mathfrak{T}_n} \right) \widehat{E}_T(d\lambda), \qquad a < 0.
\end{aligned} \tag{5.24}$$

Similarly, using representations (5.2) and (5.24) and applying Proposition 3.2 yields

$$R_n^{-1} M_n^A(a) = \frac{1}{\sqrt{(M^{S_n})'(a)}} M_n^A(a) = \int_{\Delta_n} \left(\frac{1}{\sqrt{(M^A)'(a-\lambda)}} M^A(a-\lambda) \otimes I_{\mathfrak{T}_n} \right) \widehat{E}_T(d\lambda), \qquad a < 0. \tag{5.25}$$

Setting $\mathcal{H}_{S_n} := \mathcal{H}^A \otimes \mathfrak{T}_n$,

$$\widetilde{\Gamma}_0^{S_n} = \sqrt{(M^{S_n})'(a)} \Gamma_0^{S_n} \qquad \text{and} \qquad \widetilde{\Gamma}_1^{S_n} = \frac{1}{\sqrt{(M^{S_n})'(a)}} (\Gamma_1^{S_n} - M^{S_n}(a)) \Gamma_0^{S_n}), \tag{5.26}$$

we obtain an ordinary boundary triplet $\widetilde{\Pi}_{S_n} = \{\mathcal{H}_{S_n}, \widetilde{\Gamma}_0^{S_n}, \widetilde{\Gamma}_1^{S_n}\}$ for S_n^*. Inserting formulas (5.24) and (5.23) in (5.26) yields (5.21) with Δ_n in place of $\mathbb{R}_+$. Now applying Proposition 2.8 (see formula (2.21)) one gets that the direct sum

$\widetilde{\Pi}_S := \bigoplus_{n\in\mathbb{N}} \widetilde{\Pi}_{S_n}$ is an ordinary boundary triplet for S^*. In particular, for any $f \in \mathfrak{D} = \operatorname{dom}(S^*) \cap \operatorname{dom}(A^* \otimes I_{\mathfrak{T}})$

$$\begin{aligned}\widetilde{\Gamma}_0^S f &:= \bigoplus_{n=1}^{\infty} \widetilde{\Gamma}_0^{S_n} f = \bigoplus_{n=1}^{\infty} \left(\int_{[n-1,n)} \left(\sqrt{(M^A)'(a-\lambda)} \otimes I_{\mathfrak{T}_n} \right) \widehat{E}_{T_n}(d\lambda) \right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}_n}) f \\ &= \left(\int_{\mathbb{R}_+} \left(\sqrt{(M^A)'(a-\lambda)} \otimes I_{\mathfrak{T}} \right) \widehat{E}_{T_n}(d\lambda) \right) \cdot (\Gamma_0^A \widehat{\otimes} I_{\mathfrak{T}}) f, \end{aligned} \tag{5.27}$$

which proves the first formula in (5.21). Note that convergence of the last integral for every $f \in \mathfrak{D}$ (cf. (5.4)) is guaranteed by Lemma 5.2. Formula (5.21) for $\widetilde{\Gamma}_1^S$ is proved similarly.

(ii) It easily follows from (5.26) that the Weyl function $\widetilde{M}^{S_n}(\cdot)$ corresponding to the triplet $\widetilde{\Pi}_{S_n}$ is

$$\widetilde{M}^{S_n}(z) = R_n^{-1} \left(M^{S_n}(z) - M^{S_n}(a) \right) R_n^{-1} = \frac{1}{\sqrt{(M^{S_n})'(a)}} \left(M^{S_n}(z) - M^{S_n}(a) \right) \frac{1}{\sqrt{(M^{S_n})'(a)}} \tag{5.28}$$

Inserting formulas (5.24) and (5.2) into (5.28) and applying Proposition 3.2 we arrive at the following representation

$$\widetilde{M}^{S_n}(z) = \int_{\Delta_n} \left(\frac{1}{\sqrt{(M^A)'(a-\lambda)}} \left(M^A(z-\lambda) - M^A(a-\lambda) \right) \frac{1}{\sqrt{(M^A)'(a-\lambda)}} \right) \widehat{E}_T(d\lambda), \quad z \in \mathbb{C}_\pm.$$

Finally applying Proposition 2.8 and taking notation (5.9) into account we arrive at formula for the Weyl function $\widetilde{M}^S(\cdot)$ corresponding to $\widetilde{\Pi}_S$,

$$\widetilde{M}^S(z) f = \bigoplus_1^{\infty} \widetilde{M}^{S_n}(z) f = \bigoplus_1^{\infty} \int_{\Delta_n} \left(L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{T}} \right) \widehat{E}_T(d\lambda) f = \int_{\mathbb{R}_+} \left(L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{T}} \right) \widehat{E}_T(d\lambda) f, \tag{5.29}$$

exist for every $f \in \mathcal{H}^A \otimes \mathfrak{T}$ and any $z \in \mathbb{C} \setminus \mathbb{R}_+$. Note that Lemma 5.2 ensures convergence of the last integral for every $f \in \mathcal{H}^A \otimes \mathfrak{T}$. Comparison with (5.8) proves the second equality in (5.22). The first one is extracted by combining the first formula in (2.18) with (5.24) and applying Proposition 3.2. □

B. Friedrichs and Krein extensions of $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$

In this section we assume that both a symmetric operator $A \in \mathcal{C}(\mathfrak{H})$ and the operator $T = T^*$ are non-negative. Then the set $\mathrm{Ext}_{\mathrm{A}}[0,\infty)$ of non-negative self-adjoint extensions of A is non-empty (see[1,7,21]). Moreover, according to the Krein result[24] the set $\mathrm{Ext}_{\mathrm{A}}[0,\infty)$ contains two extremal extensions: a maximal non-negative extension $\widehat{A}_{\mathrm{F}}$ (also called *Friedrichs'* or *hard* extension) and a minimal non-negative extension $\widehat{A}_{\mathrm{K}}$ (*Krein's* or *soft* extension). The latter are uniquely determined by the following inequalities

$$(\widehat{A}_F + x)^{-1} \le (\widetilde{A} + x)^{-1} \le (\widehat{A}_K + x)^{-1}, \qquad x \in (0,\infty), \quad \widetilde{A} \in \mathrm{Ext}_{\mathrm{A}}(0,\infty)$$

(for detail we refer the reader to[1,21]).

Recall the following statements.

Proposition 5.4 ([16]) *Let $A \ge 0$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 (= A^* \upharpoonright \ker \Gamma_0) \ge 0$. Let $M(\cdot)$ be the corresponding Weyl function. Then $A_0 = \widehat{A}_{\mathrm{F}}$ $(A_0 = \widehat{A}_{\mathrm{K}})$ if and only if*

$$\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty, \qquad \big(\lim_{x \uparrow 0} (M(x)f, f) = +\infty \big), \qquad f \in \mathcal{H} \setminus \{0\}. \tag{5.30}$$

It is known that under the conditions of Proposition 5.4 the following implication holds $\widetilde{A} = \widetilde{A}^* = A_\Theta$ is semi-bounded below $\Longrightarrow$ Θ is semi-bounded below while the equivalence does not hold in general.

Definition 5.5 *Let $A \ge 0$ be a non-negative symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 = \widetilde{A}_F$. We say that A satisfies LSB-property (abbreviation of lower semi-boundedness) if the following equivalence holds:*

$$A_\Theta = A_\Theta^* \textit{ is lower semi-bounded} \Longleftrightarrow \Theta = \Theta^* \textit{ is lower semi-bounded.}$$

To describe the operators with LSB-property we introduce the following definition.

It is said that $M(\cdot)$ uniformly tends to $-\infty$ (in symbols $M(\cdot) \rightrightarrows -\infty$) if for any $N > 0$ there exists x_N such that

$$\big(M(x)h, h\big) \leq -N \cdot \|h\|^2 \qquad \text{for } x \leq x_N, \quad h \in \mathcal{H}. \tag{5.31}$$

Clearly, (5.31) implies (5.30) but not vice versa.

Proposition 5.6 ([16]) *Let $A \geq 0$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 = \widetilde{A}_F$. Then the following statements are equivalent:*

(i) A satisfies LSB property;

(ii) $M(x) \rightrightarrows -\infty$ as $x \to -\infty$.

Further we describe the Friedrichs extension $\widehat{S}_F$ of S by means of the extensions $\widehat{A}_F$ of A and investigate the problem of semi-boundedness of extensions of S.

We start with the following simple algebraic lemma.

Lemma 5.7 *Let $\{X_k\}_1^n$ be a sequence of positive definite operators in $\mathcal{H}$, $X_k \geq dI_{\mathcal{H}} > 0$, and let $E_T(\cdot)$ be a spectral measure of the operator $T = T^* \in [\mathfrak{T}]$. Then for any partition $\{\Delta_k\}_1^n$ of $[a,b]$ $\big(\sigma(T) \subset [a,b]\big)$ one has*

$$X := \sum_k X_k \otimes E_T(\Delta_k) \geq d. \tag{5.32}$$

Proof. Since $X_k \geq dI_{\mathcal{H}} > 0$, the operator $(X_k - dI_{\mathcal{H}}) \otimes E_T(\Delta_k)$ is non-negative. Hence

$$X = \sum_k X_k \otimes E_T(\Delta_k) \geq d \sum_k I_{\mathcal{H}} \otimes E_T(\Delta_k) = dI_{\mathcal{H}} \otimes \left(\sum_k E_T(\Delta_k)\right) = dI_{\mathcal{H}} \otimes I_{\mathfrak{T}} = dI_{\mathfrak{R}}. \tag{5.33}$$

□

Proposition 5.8 *Let A be a non-negative symmetric operator in $\mathfrak{H}, T = T^* \geq 0$ and $S := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. Then:*

$$\widehat{S}_F = \widehat{A}_F \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T \qquad \textit{and} \qquad \widehat{S}_K = \widehat{A}_K \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T. \tag{5.34}$$

Proof. (i) Assume for the beginning that T is bounded, $T \in [\mathfrak{T}]$. Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* such that $A_0 = \widehat{A}_F$. Then, by Theorem 5.1, $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\} := \Pi_A \widehat{\otimes} I_{\mathfrak{T}}$ is a boundary triplet for S^* satisfying $S_0 := S^* \restriction \ker(\Gamma_0^S) = A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$, and the corresponding Weyl function $M^S(\cdot)$ is given by (5.2).

To prove the first relation in (5.34) it suffices to check condition (5.30) for $M^S(\cdot)$. Let $h := \sum_{j=1}^n h_j' \otimes h_j''$ where $h_j' \in \mathcal{H}^A, h_j'' \in \mathfrak{T}$, let $\mathcal{H}_n^A := \text{span}\{h_j' : 1 \leq j \leq n\}$ and let P_n be the orthogonal projection on $\mathcal{H}_n^A$ in $\mathcal{H}^A$.

Since $A_0 = \widehat{A}_F$, the Weyl function $M^A(\cdot)$ satisfies condition (5.30). Setting $M_n^A(\cdot) = P_n M(\cdot) \restriction \mathcal{H}_n^A$ we note that due to the compactness of the finite-dimensional ball condition (5.30) is uniform on each $\mathcal{H}_n^A$. In other words, for each $N > 0$ there exists $x_N < 0$ such that

$$-M_n^A(x) \geq N \qquad \text{for} \qquad x \leq x_N. \tag{5.35}$$

Since $A_0 \geq 0$, Theorem 5.1 ensures that the Weyl function $M^A(\cdot)$ being a holomorphic in $\mathbb{C} \setminus \mathbb{R}_+$ admits the integral representation (5.2) for any $z = x < 0$ and $\lambda > 0$. Let $\pi = \{\Delta_k\}_1^p$ be a partition of $\Delta = [a,b]$, let $\lambda_k \in \Delta_k$, and let

$$S_p(\pi) = \sum_{k=1}^p M^A(x_N - \lambda_k) \otimes E_T(\Delta_k) \tag{5.36}$$

be an integral sum for the integral (5.2) with $x = x_N$. Setting $Y_k = M_n^A(x_N - \lambda_k)$, $k \in \{1, \ldots, p\}$, one gets

$$\begin{aligned}(P_n \otimes I_{\mathfrak{T}})S_p(\pi)h &= \sum_k \sum_j P_n M^A(x_N - \lambda_k)h_j' \otimes E_T(\Delta_k)h_j'' \\ &= \sum_k \sum_j Y_k h_j' \otimes E_T(\Delta_k)h_j'' = \sum_k (Y_k \otimes E_T(\Delta_k))h. \end{aligned} \tag{5.37}$$

Combining this relation with (5.35) and noting that $h \in \mathcal{H}_n^A \otimes \mathfrak{T}$ and $x_N - \lambda_k < x_N$ one gets from Lemma 5.7 that

$$\big(S_p(\pi)h, h\big) = \big((P_n \otimes I_{\mathfrak{T}})S_p(\pi)h, h\big) \le -N \tag{5.38}$$

Passing here to the limit as the diameter $|\pi|$ of partition π tends to zero and taking formula (5.2) for the Weyl function into account and setting $M_n^S(\cdot) = (P_n \otimes I_{\mathfrak{T}})M(\cdot) \upharpoonright \mathcal{H}_n^A \otimes I_{\mathfrak{T}}$, one derives

$$\big(M^S(x)h, h\big) = \big(M_n^S(x)h, h\big) \le -N \qquad \text{for} \qquad x \le x_N. \tag{5.39}$$

Since finite tensors $h = \sum_{j=1}^n h_j' \otimes h_j''$ are dense in $\mathcal{H}^A \otimes \mathfrak{T}$, this inequality yields condition (5.30) for $M(\cdot) = M^S(\cdot)$ and arbitrary $h \in \mathcal{H}^A \otimes \mathfrak{T}$.

(ii) Let $T \in \mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H})$. Then T admits a decomposition

$$T = \bigoplus_1^\infty T_n, \quad \text{where} \quad T_n := TE_T[n-1, n) \in \mathcal{B}(\mathcal{H}_n), \quad \text{and} \quad \mathcal{H}_n := E_T[n-1, n)\mathcal{H}. \tag{5.40}$$

Hence

$$S = \bigoplus_1^\infty S_n \qquad \text{where} \quad S_n := A \otimes I_{\mathcal{H}_n} + I_{\mathfrak{H}} \otimes T_n. \tag{5.41}$$

Clearly, S_n is a non-negative symmetric operator in $\mathfrak{H} \otimes \mathcal{H}_n$. According to[26] (Corollary 3.10)

$$\widehat{S}_F = \bigoplus_1^\infty \widehat{S}_{n,F} \qquad \text{and} \qquad \widehat{S}_K = \bigoplus_1^\infty \widehat{S}_{n,K}, \tag{5.42}$$

where $\widehat{S}_{n,F}$ and $\widehat{S}_{n,K}$ denote the Friedrichs' and Krein's extensions of the symmetric non-negative operator S_n, respectively. Combining representations (5.42) with representations (5.34) with bounded $T_n \in \mathcal{B}(\mathcal{H}_n)$ in place of $T \in \mathcal{B}(\mathcal{H})$ proved at the previous step, implies

$$\widehat{S}_F = \bigoplus_1^\infty \widehat{S}_{n,F} = \bigoplus_1^\infty (\widehat{A}_F \otimes I_{\mathcal{H}_n} + I_{\mathfrak{H}} \otimes T_n) = \widehat{A}_F \otimes I_{\mathcal{H}} + \bigoplus_1^\infty (I_{\mathfrak{H}} \otimes T_n) = \widehat{A}_F \otimes I_{\mathcal{H}} + I_{\mathfrak{H}} \otimes T. \tag{5.43}$$

The representation for S_K is proved similarly. □

Proposition 5.9 *Let A be a non-negative symmetric operator in $\mathfrak{H}$ and let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0^A) = \widehat{A}_F$. Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Let also $T = T^* \in [\mathfrak{T}]$, $T \ge 0$ and let $S = A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. If A satisfies the LSB-property, then the operator S also satisfies the LSB–property.*

Proof. Consider a boundary triplet $\widetilde{\Pi}_S = \{\widetilde{\mathcal{H}}^S, \widetilde{\Gamma}_0^S, \widetilde{\Gamma}_1^S\}$ for S^* given by (5.21). By Theorem 5.1(i),

$$S_0 = S^* \upharpoonright \ker(\Gamma_0^S) = \widehat{S}_F = \widehat{A}_F \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T.$$

Since A satisfies the LSB-property and $A_0 = \widehat{A}_F$, Theorem 5.6 ensures that the Weyl function $M^A(\cdot)$ tends to $-\infty$ uniformly, i.e. $M^A(x) \rightrightarrows -\infty$ as $x \to -\infty$. In other words, for each $N > 0$ there exists $x_N < 0$ such that $-M^A(x) \ge N \quad$ for $\quad x \le x_N$.

By Theorem 5.1(i) the Weyl function $M^S(\cdot)$ corresponding to Π_S is given by (5.2). Let $\pi = \{\Delta_k\}_1^p$ be a partition of $\Delta = [a, b]$ and let $\lambda_k \in \Delta_k$. Then applying Lemma 5.7 to the integral sum (5.36) we get

$$-S_p(\pi) = -\sum_{k=1}^p M^A(x_N - \lambda_k) \otimes E_T(\Delta_k) \ge N. \tag{5.44}$$

Passing here to the limit as $|\pi| \to 0$ one obtains

$$-M^S(x) = \int_\Delta \widehat{E}_T(d\lambda)\left(M^A(z-\lambda) \otimes I_{\mathfrak{T}}\right) \ge N \qquad \text{for} \qquad x \le x_N.$$

The latter amounts to saying that $M^S(x) \rightrightarrows -\infty$ as $x \to -\infty$. By Theorem 5.6 this property implies (in fact is equivalent to) the LSB-property of S. □

6. EXAMPLES

In what follows the operator T is arbitrary self-adjoint operator acting on a separable Hilbert space $\mathfrak{T}$.

A. Schrödinger operators and bosons in 1D

1. *Schrödinger operators on half-lines*

Let $v_r \in \mathbb{R}$, $b \in \mathbb{R}$, and let $H_r = -\frac{d^2}{dx^2} + v_r$ denote a minimal operator in $\mathfrak{H}_r := L^2(\Delta_r)$, $\Delta_r = (b, \infty)$. Clearly, $\mathrm{dom}\,(H_r) = W^{2,2}_{00}(\Delta_r) := \{f \in W^2_2((b,\infty)) : f(b) = f'(b) = 0\}$ and H_r is a closed densely defined symmetric operator with $n_\pm(H_r) = 1$. The adjoint operator is given by the same expression $H_r^* = -\frac{d^2}{dx^2} + v_r$ on the domain $\mathrm{dom}\,(H_r^*) = W^{2,2}(\Delta_r)$. One easily checks that a triplet $\Pi_{H_r} = \{\mathcal{H}^{H_r}, \Gamma_0^{H_r}, \Gamma_1^{H_r}\}$ with

$$\mathcal{H}^{H_r} := \mathbb{C}, \quad \Gamma_0^{H_r} f = f(b), \quad \text{and} \quad \Gamma_1^{H_r} f = f'(b), \quad f \in \mathrm{dom}\,(H_r{}^*),$$

is a boundary triplet for H_r^*. The corresponding γ-field $\gamma^{H_r}(\cdot)$ and Weyl function $M^{H_r}(\cdot)$ are given by

$$(\gamma^{H_r}(z)\xi)(x) = e^{i\sqrt{z-v_r}(x-b)}\xi, \quad \xi \in \mathbb{C}, \quad x \in \Delta_r, \quad z \in \mathbb{C}_\pm,$$

and

$$M^{H_r}(z) = m^{H_r}(z) = i\sqrt{z - v_r}, \quad z \in \mathbb{C}_\pm,$$

respectively. The function $\sqrt{\cdot}$ is defined on $\mathbb{C}$ with the cut along the positive semi-axis $\mathbb{R}_+$. Its branch is fixed by the condition $\sqrt{1} = 1$. Clearly, the Weyl function $M^{H_r}(\cdot)$ is a scalar function.

Let us consider the closed densely defined symmetric operator

$$S_r = \overline{H_r \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_r} \otimes T} \tag{6.1}$$

on the Hilbert space $\mathfrak{K}_r := \mathfrak{H}_r \otimes \mathfrak{T} = L_2(\Delta_r, \mathfrak{T})$. In the following we use the notation $\vec{f}(x)$, $x \in \Delta_r$ for elements of $\mathfrak{K}_r = L_2(\Delta_r, \mathfrak{T})$. In accordance with Theorem 4.8 there is a boundary triplet $\Pi_{K_r} = \{\mathcal{H}^{S_r}, \Gamma_0^{S_r}, \Gamma_1^{S_r}\}$ for $S_r{}^*$ such that $\mathcal{H}^{S_r} = \mathcal{H}^{H_r} \otimes \mathfrak{T} = \mathfrak{T}$,

$$\begin{aligned}
\Gamma_0^{S_r}\vec{f} &= \sqrt{\mathrm{Im}(m^{H_r}(i-T))}\,\vec{f}(b),\\
\Gamma_1^{S_r}\vec{f} &= \frac{1}{\sqrt{\mathrm{Im}(m^{H_r}(i-T))}}\left(\vec{f}'(b) - \mathrm{Re}(m^{H_r}(i-T))\vec{f}(b)\right),
\end{aligned} \tag{6.2}$$

$\vec{f} \in \mathrm{dom}\,(H_r{}^* \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{H}_r} \otimes T) = W^{2,2}(\Delta_r, \mathfrak{T}) \cap \mathrm{dom}\,(I_{\mathfrak{H}_r} \otimes T) \subseteq \mathrm{dom}\,(K_r^*)$. The corresponding γ-field $\gamma^{S_r}(\cdot) : \mathfrak{T} \longrightarrow \mathfrak{K}_r$ and Weyl function $M^{S_r}(\cdot) : \mathfrak{T} \longrightarrow \mathfrak{T}$ are given by

$$(\gamma^{S_r}\xi)(x) = e^{i\sqrt{z-v_r-T}(x-b)}\frac{1}{\sqrt{\mathrm{Im}(m^{H_r}(i-T))}}\xi, \quad \xi \in \mathfrak{T}, \quad x \in \Delta_r,$$

and

$$M^{S_r}(z) = \frac{m^{H_r}(z-T) - \mathrm{Re}(m^{H_r}(i-T))}{\mathrm{Im}(m^{H_r}(i-T))}, \qquad z \in \mathbb{C}_\pm.$$

Of course, the considerations are similar for the interval $\Delta_l = (-\infty, a)$, $a \in \mathbb{R}$. Let $H_l = -\frac{d^2}{dx^2} + v_l$, $v_l \in \mathbb{R}$, with domain $\mathrm{dom}\,(H_l) := W^{2,2}_{00}(\Delta_l)$ defined on $\mathfrak{H}_l := L^2(\Delta_l, \mathfrak{T})$. One checks that $\Pi_{H_l} = \{\mathcal{H}^{H_l}, \Gamma_0^{H_l}, \Gamma_1^{H_l}\}$,

$$\mathcal{H}^{H_l} := \mathbb{C}, \quad \Gamma_0^{H_l} f = f(a), \quad \text{and} \quad \Gamma_1^{H_l} f = -f'(a), \quad f \in \mathrm{dom}\,(H_l{}^*),$$

is a boundary triplet for H_l^*. The Gamma field and the Weyl function are computed by

$$(\gamma^{H_l}(z)\xi)(x) = e^{i\sqrt{z-v_l}(a-x)}\xi, \quad \xi \in \mathbb{C}, \quad x \in \Delta_l, \quad z \in \mathbb{C}_\pm,$$

and

$$M^{H_l}(z) = m^{H_l}(z) = i\sqrt{z - v_l}, \quad z \in \mathbb{C}_\pm.$$

Let us consider the closed densely defined symmetric operator $S_l = \overline{H_l \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_l} \otimes T}$ acting in $\mathfrak{K}_l := \mathfrak{H}_l \otimes \mathfrak{T} = L^2(\Delta_l, \mathfrak{T})$. As above one finds

$$\begin{aligned} \Gamma_0^{S_l} \vec{f} &= \sqrt{\operatorname{Im}(m^{H_l}(i-T))}\, \vec{f}(a), \\ \Gamma_1^{S_l} f &= \frac{1}{\sqrt{\operatorname{Im}(m^{H_l}(i-T))}} \left(-\vec{f}'(a) - \operatorname{Re}(m^{H_l}(i-T))\vec{f}(a)\right) \end{aligned} \tag{6.3}$$

$\vec{f} \in \operatorname{dom}(H_l{}^* \otimes I_{\mathfrak{T}}) \cap \operatorname{dom}(I_{\mathfrak{H}_l} \otimes T) = W^{2,2}(\Delta_l, \mathfrak{T}) \cap \operatorname{dom}(I_{\mathfrak{H}_l} \otimes T) \subseteq \operatorname{dom}(S_l^*)$ as well as

$$(\gamma^{S_l}\xi)(x) = e^{i\sqrt{z - v_l - T}(a-x)} \frac{1}{\sqrt{\operatorname{Im}(m^{H_l}(i-T))}}\xi, \quad \xi \in \mathfrak{T}, \quad x \in \Delta_r,$$

and

$$M^{S_l}(z) = \frac{m^{H_l}(z-T) - \operatorname{Re}(m^{H_l}(i-T))}{\operatorname{Im}(m^{H_l}(i-T))}, \qquad z \in \mathbb{C}_\pm.$$

2. Schrödinger operators on bounded intervals

Let $\Delta_c = (a,b)$ and $v_c \in \mathbb{R}$. Consider a minimal Sturm-Liouville operator H_c in $\mathfrak{H}_c = L^2(\Delta_c)$ given by

$$\begin{aligned} (H_c f)(x) &= -\frac{d^2}{dx^2} f(x) + v_c f(x), \quad x \in \Delta_c, \\ f \in \operatorname{dom}(H_c) &= \left\{ f \in W^{2,2}(\Delta_c) : \begin{array}{c} f(a) = f(b) = 0 \\ f'(a) = f'(b) = 0 \end{array} \right\}. \end{aligned}$$

Clearly, H_c is a closed symmetric operator with the deficiency indices $n_\pm(A) = 2$. Its adjoint H_c^* is given by

$$(H_c^* f)(x) = -\frac{d^2}{dx^2} f(x) + v_c f(x), \quad f \in \operatorname{dom}(H_c^*) = W^{2,2}(\Delta_c).$$

Consider the extension (Dirichlet realization) H_c^D of the minimal operator H_c defined by

$$H_c^D = -\frac{d^2}{dx^2} + v_c, \qquad \operatorname{dom}(H_c^D) = \{f \in W^{2,2}(\Delta_c) : f(a) = f(b) = 0\}.$$

The Neumann extension (realization) H_N is fixed by

$$H_c^N = -\frac{d^2}{dx^2} + v_c, \quad \operatorname{dom}(H_c^N) = \{f \in W^{2,2}(\Delta_c) : f'(a) = f'(b) = 0\}.$$

One easily checks that the triplet $\Pi_{H_c} := \{\mathcal{H}^{H_c}, \Gamma_0^{H_c}, \Gamma_1^{H_c}\}$ with

$$\mathcal{H}^{H_c} := \mathbb{C}^2, \quad \Gamma_0^{H_c} f = \frac{1}{\sqrt{2}} \begin{pmatrix} f(a) + f(b) \\ f(a) - f(b) \end{pmatrix}, \quad \Gamma_1^{H_c} f = \frac{1}{\sqrt{2}} \begin{pmatrix} f'(a) - f'(b) \\ f'(a) + f'(b) \end{pmatrix},$$

$f \in \operatorname{dom}(H_c^*)$, is a boundary triplet for H_c^*. Clearly, $H_c^D = H_c^* \upharpoonright \ker(\Gamma_0^{H_c})$ and $H_c^N = H_c^* \upharpoonright \ker(\Gamma_1^{H_c})$. The corresponding γ-field $\gamma^{H_c}(\cdot)$ and Weyl function $M^{H_c}(\cdot)$ are given by

$$(\gamma^{H_c}(z)\xi)(x) = \frac{1}{\sqrt{2}} \left(\frac{\cos(\sqrt{z - v_c}(x-\nu))}{\cos(\sqrt{z - v_c}\, d)}, -\frac{\sin(\sqrt{z - v_c}(x-\nu))}{\sin(\sqrt{z - v_c}\, d)} \right) \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

$z \in \mathbb{C}_\pm$, $x \in (a,b)$, $\nu := \frac{a+b}{2}$, $d := \frac{b-a}{2}$, and

$$M^{H_c}(z) = \begin{pmatrix} m_1^{H_c}(z) & 0 \\ 0 & m_2^{H_c}(z) \end{pmatrix}, \quad z \in \mathbb{C}_\pm,$$

where

$$\begin{aligned} m_1^{H_c}(z) &:= \sqrt{z-v_c}\tan(\sqrt{z-v_c}\,d), \\ m_2^{H_c}(z) &:= -\sqrt{z-v_c}\cot(\sqrt{z-v_c}\,d), \end{aligned} \qquad z \in \mathbb{C}_\pm.$$

Notice that the Weyl function $M^{H_c}(\cdot)$ is of quasi scalar type.

We consider the closed densely defined symmetric operator

$$S_c := \overline{H_c \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_H} \otimes T}. \tag{6.4}$$

defined on $\mathfrak{K}_c := \mathfrak{H}_c \otimes \mathfrak{T} = L^2(\Delta_c, \mathfrak{T})$. Elements of $L^2(\Delta_c, \mathfrak{T})$ are denoted by $\vec{f}(x)$, $x \in \Delta_c$. Obviously, the self-adjoint operators $S_c^D := \overline{H_c^D \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_H} \otimes T}$ and $S_c^N := \overline{H_c^N \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}_H} \otimes T}$ are self-adjoint extensions of K_c.

Let us introduce the subspaces $\mathcal{H}_1^{H_c} := \mathbb{C}$ and $\mathcal{H}_2^{H_c} := \mathbb{C}$. Notice that $\mathcal{H}^{H_c} = (\mathcal{H}_1^{H_c} \oplus \mathcal{H}_2^{H_c})^t$. It follows from (4.54) that there is a boundary triplet $\Pi_{S_c} = \{\mathcal{H}^{S_c}, \Gamma_0^{S_c}, \Gamma_1^{S_c}\}$ for S_c^* such that

$$\mathcal{H}^{S_c} = \mathcal{H}^{H_c} \otimes \mathfrak{T} = \begin{matrix} \mathcal{H}_1^{H_c} \otimes \mathfrak{T} \\ \oplus \\ \mathcal{H}_2^{H_c} \otimes \mathfrak{T} \end{matrix} = \begin{matrix} \mathfrak{T} \\ \oplus \\ \mathfrak{T} \end{matrix} =: \begin{matrix} \mathcal{H}_1^{K_c} \\ \oplus \\ \mathcal{H}_2^{K_c} \end{matrix}$$

and

$$\begin{aligned} &\Gamma_0^{S_c}\vec{f} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\mathrm{Im}(m_1^{H_c}(i-T)}(\vec{f}(a) + \vec{f}(b) \\ \sqrt{\mathrm{Im}(m_2^{H_c}(i-T))}(\vec{f}(a) - \vec{f}(b)) \end{pmatrix} \\ &\Gamma_1^{S_c}(z)\vec{f} = \\ &\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\mathrm{Im}(m_1^{H_c}(i-T)}} \left(\vec{f}'(a) - \vec{f}'(b) - \mathrm{Re}(m_1^{H_c}(i-T))(\vec{f}(a) + \vec{f}(b)\right) \\ \frac{1}{\sqrt{\mathrm{Im}(m_2^{H_c}(i-T))}} \left(\vec{f}'(a) + \vec{f}'(b)) - \mathrm{Re}(m_2^{H_c}(i-T))(\vec{f}(a) - \vec{f}(b))\right) \end{pmatrix} \end{aligned}$$

$\vec{f} \in \mathrm{dom}\,(H_c^* \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{H}_c} \otimes T) = W^{2,2}(\Delta_c, \mathfrak{T}) \cap \mathrm{dom}\,(I_{\mathfrak{H}_c} \otimes T) \subseteq \mathrm{dom}\,(S_c^*)$. From (4.55) we get the Gamma field $\gamma^{S_c}(\cdot) : (\mathcal{H}_1^{S_c} \oplus \mathcal{H}_2^{S_c})^t \longrightarrow \mathfrak{K}_c$,

$$\begin{aligned} (\gamma^{S_c}(z)\vec{\xi})(x) = &\frac{\cos(\sqrt{z-T-v_c}(x-\nu))}{\sqrt{2}\cos(\sqrt{z-T-v_c}\,d)\sqrt{\mathrm{Im}(m_1^{H_c}(i-T))}}\vec{\xi}_1 \\ &- \frac{\sin(\sqrt{z-T-v_c}(x-\nu))}{\sqrt{2}\sin(\sqrt{z-T-v_c}\,d)\sqrt{\mathrm{Im}(m_2^{H_c}(i-T))}}\vec{\xi}_2, \end{aligned}$$

$z \in \mathbb{C}_\pm$. Finally, from (4.56) the Weyl function $M^{S_c}(\cdot) : (\mathcal{H}_1^{S_c} \oplus \mathcal{H}_2^{S_c})^t \longrightarrow (\mathcal{H}_1^{S_c} \oplus \mathcal{H}_2^{S_c})^t$ is computed by

$$M^{S_c}(z) = \begin{pmatrix} \frac{m_1^{H_c}(z-T) - \mathrm{Re}(m_1^{H_c}(i-T))}{\mathrm{Im}(m_1^{H_c}(i-T))} & 0 \\ 0 & \frac{m_2^{H_c}(z-T) - \mathrm{Re}(m_2^{H_c}(i-T))}{\mathrm{Im}(m_2^{H_c}(i-T))} \end{pmatrix}, \qquad z \in \mathbb{C}_\pm.$$

Remark 6.1 Sturm-Liouville operators S_c with operator-valued potential $T = T^* \in \mathcal{C}(\mathfrak{T})$ have first been treated on a finite interval in the pioneering paper by M.L. Gorbachuk[19]. Clearly, the corresponding minimal operator S_c admits representation (6.4). In particular, a boundary triplet for S_c^* was first constructed in[19] (see also[20]). A construction of a boundary triplet for S_r^* in the case of semi-axis has first been proposed in[16] (Section 9). However, our construction (6.2) of the boundary triplet for S_r^* is borrowed from[26] where the regularization procedure was first proposed and applied to the operator S_r.

After appearance of the work[19] the spectral theory of self-adjoint and dissipative extensions of S_c in $L^2(\Delta_c, \mathfrak{T})$ has intensively been investigated. The results are summarized in[20] (Chapter 4) where one finds, in particular, criteria for discreteness of the spectra, asymptotic formulas for the eigenvalues, resolvent comparability results, etc. Spectral theory of self-adjoint extensions of S_r can be found in[26]. In particular, a criterion in order that all self-adjoint extensions of the operator S_r have absolutely continuous non-negative part is also obtained there.

B. Dirac operators and bosons in 1D

In the following we consider the Dirac operator instead of the Schrödinger operator, cf.[8].

1. *Dirac operators on half-lines*

In the Hilbert space $\mathfrak{D}_r = L^2(\Delta_r, \mathbb{C}^2)$, $\Delta_r = (b, \infty)$, let us consider the Dirac operator

$$(D_r f)(x) := -ic\frac{d}{dx} \otimes \sigma_1 f(x) + \frac{c^2}{2} \otimes \sigma_3 f(x), \quad x \in \Delta_r,$$
$$f \in \mathrm{dom}\,(D_r) := W^{1,2}_0(\Delta_r, \mathbb{C}^2) := \{f \in W^{1,2}(\Delta_r, \mathbb{C}^2) : f(b) = 0\}.$$

Here

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that

$$(D^*_r f)(x) = -ic\frac{d}{dx} \otimes \sigma_1 f(x) + \frac{c^2}{2} \otimes \sigma_3 f(x), \quad x \in \Delta_r,$$
$$f \in \mathrm{dom}\,(D^*_r) = W^{1,2}(\Delta_r, \mathbb{C}^2).$$

One easily checks that $n_\pm(D_r) = 1$. From Lemma 3.3 of[11] we get that $\Pi_{D_r} = \{\mathcal{H}^{D_r}, \Gamma^{D_r}_0, \Gamma^{D_r}_1\}$,

$$\mathcal{H}^{D_r} := \mathbb{C}, \quad \Gamma^{D_r}_0 f := f_1(b), \quad \Gamma^{D_r}_1 f := icf_2(b)$$

is a boundary triplet for D^*_r. The Gamma field $\gamma^{D_r}(\cdot)$ and Weyl function are given by

$$(\gamma^{D_r}(z)\xi)(x) = \begin{pmatrix} e^{ik(z)(x-b)}\xi \\ k_1(z)e^{ik(z)(x-b)}\xi \end{pmatrix} \quad x \in \Delta_r, \quad z \in \mathbb{C}_\pm, \quad \xi \in \mathcal{H}^{D_r}.$$

and

$$M^{D_r}(z) = m^{D_r}(z) I_{\mathcal{H}^{D_r}}, \quad m_r(z) := ic\, k_1(z), \quad z \in \mathbb{C}_\pm.$$

Here

$$k(z) := \frac{1}{c}\sqrt{z^2 - \frac{c^4}{4}}, \quad z \in \mathbb{C},$$

where the branch of the multifunction $k(\cdot)$ is fixed by the condition $k(x) > 0$ for $x > \frac{c^2}{2}$. Notice that $k(\cdot)$ is holomorphic on $\mathbb{C} \setminus \left\{(-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)\right\}$. Further,

$$k_1(z) := \frac{c\,k(z)}{z + \frac{c^2}{2}}, \quad z \in \mathbb{C}.$$

which is also holomorphic on $\mathbb{C} \setminus \left\{(-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)\right\}$. The function $k_1(\cdot)$ admits the representation

$$k_1(z) = \sqrt{\frac{z - \frac{c^2}{2}}{z + \frac{c^2}{2}}}, \quad z \in \mathbb{C},$$

where the branch of $\sqrt{\frac{z-\frac{c^2}{2}}{z+\frac{c^2}{2}}}$ is fixed by the condition $\sqrt{\frac{x-\frac{c^2}{2}}{x+\frac{c^2}{2}}} > 0$ for $x > \frac{c^2}{2}$.

Let us consider the closed densely defined symmetric operator

$$S_r = \overline{D_r \otimes I_{\mathfrak{T}} + I_{\mathfrak{D}_r} \otimes T},$$

which is defined on

$$\mathfrak{K}_r := \mathfrak{D}_r \otimes \mathfrak{T} = L^2(\Delta_r, (\mathfrak{T} \oplus \mathfrak{T})^t).$$

In the following we denote elements of $\mathfrak{K}_r$ by $\vec{f} = (\vec{f}_1, \vec{f}_2)^t$. Since D_r is not semi-bounded from below the sum $\overline{D_r \otimes I_{\mathfrak{T}} + I_{\mathfrak{D}_r} \otimes T}$ is also not semi-bounded from below. Nevertheless, by Theorem 4.8 (iii) $\Pi_{S_r} = \{\mathcal{H}^{S_r}, \Gamma_0^{S_r}, \Gamma_1^{S_r}\}$,

$$\mathcal{H}^{S_r} = \mathcal{H}^{D_r} \otimes \mathfrak{T} = \mathfrak{T}$$

and

$$\begin{aligned}
\Gamma_0^{S_r} \vec{f} &= \sqrt{\mathrm{Im}(m^{D_r}(i-T))}\, \vec{f}_1(b),\\
\Gamma_1^{S_r} \vec{f} &= \frac{1}{\sqrt{\mathrm{Im}(m^{D_r}(i-T))}} \left(ic\vec{f}_2(b) - \mathrm{Re}(m_1^{D_r}(i-T))\vec{f}_1(b) \right),
\end{aligned}$$

$\vec{f} \in \mathrm{dom}\,(D_r^* \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{D}_r} \otimes T) = W^{2,2}(\Delta_r, (\mathfrak{T} \oplus \mathfrak{T})^t) \cap \mathrm{dom}\,(I_{\mathfrak{D}_r} \otimes T) \subseteq \mathrm{dom}\,(S_r^*)$, defines a boundary triplet for S_r^*. The Gamma field $\gamma^{L_r}(\cdot) : \mathfrak{T} \longrightarrow \mathfrak{K}_r$ and Weyl function $M^{S_r}(\cdot) : \mathfrak{T} \longrightarrow \mathfrak{T}$ are given by

$$(\gamma^{S_r}(z)\xi)(x) = \begin{pmatrix} e^{ik(z-T)(x-b)} \frac{1}{\mathrm{Im}(m^{D_r}(i-T))}\xi \\ k_1(z-T)e^{ik(z-T)(x-b)} \frac{1}{\mathrm{Im}(m^{D_r}(i-T))}\xi \end{pmatrix}, \quad \xi \in \mathfrak{T},$$

$x \in \Delta_r$, $z \in \mathbb{C}_\pm$ and

$$M^{S_r}(z) = \frac{m^{D_r}(z-T) - \mathrm{Re}(m^{D_r}(i-T))}{\mathrm{Im}(m^{D_r}(i-T))}, \quad z \in \mathbb{C}_\pm.$$

The Dirac operator on the half-axis $(-\infty, a)$ can be treated in the same way.

2. Dirac operators on bounded intervals

Let us consider the closed densely defined symmetric operator

$$\begin{aligned}
(D_c f)(x) &:= -ic\frac{d}{dx} \otimes \sigma_1 f(x) + \frac{c^2}{2} \otimes \sigma_3 f(x), \quad x \in \Delta_c,\\
f \in \mathrm{dom}\,(D_c) &:= W_0^{1,2}(\Delta_c, \mathbb{C}^2) := \{f \in W^{1,2}(\Delta_c, \mathbb{C}^2) : f(a) = f(b) = 0\},
\end{aligned}$$

where $\Delta_c = (a, b)$, acting in the Hilbert space $\mathfrak{D}_c := L^2(\Delta_c, \mathbb{C}^2)$. Notice that $n_\pm(D_c) = 2$. The adjoint operator D_c^* looks like

$$\begin{aligned}
(D_c^* f)(x) &= -ic\frac{d}{dx} \otimes \sigma_1 f(x) + \frac{c^2}{2} \otimes \sigma_3 f(x), \quad x \in \Delta_c,\\
f \in \mathrm{dom}\,(D_c^*) &= W^{1,2}(\Delta_c, \mathbb{C}^2).
\end{aligned}$$

The triplet $\Pi_{D_c} = \{\mathcal{H}^{D_c}, \Gamma_0^{D_c}, \Gamma_1^{D_c}\}$, $\mathcal{H}^{D_c} := \mathbb{C}^2$,

$$\begin{aligned}
\Gamma_0^{D_c} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &:= \frac{1}{\sqrt{2}} \begin{pmatrix} f_1(a) + f_1(b) \\ f_1(a) - f_1(b) \end{pmatrix},\\
\Gamma_1^{D_c} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &:= \frac{ic}{\sqrt{2}} \begin{pmatrix} f_2(a) - f_2(b) \\ f_2(a) + f_2(b) \end{pmatrix},
\end{aligned}$$

$f \in \mathrm{dom}\,(D_c^*)$, forms a boundary triplet for D_c^*. The Gamma field and the Weyl function are given by

$$\gamma^{D_c}(z) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\cos(k(z)(x-\nu))}{\cos(k(z)d)} & \frac{\sin(k(z)(x-\nu))}{\sin(k(z)d)} \\ ik_1(z)\frac{\sin(k(z)(x-\nu))}{\cos(k(z)d} & ik_1(z)\frac{\cos(k(z)(x-\nu))}{\sin(k(z)d)} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

$z \in \mathbb{C}_\pm$, and

$$M^{D_c}(z) = \begin{pmatrix} m_1^{D_c}(z) & 0 \\ 0 & m_2^{D_c}(z) \end{pmatrix}, \quad z \in \mathbb{C}_\pm,$$

where

$$\begin{array}{l} m_1^{D_c}(z) := ck_1(z)\tan(k(z)d) \\ m_2^{D_c}(z) := -ck_1(z)\cot(k(z)d) \end{array}, \quad z \in \mathbb{C}_\pm,$$

and $d := \frac{b-a}{2}$, $\nu := \frac{b+a}{2}$. Notice that the Weyl function $M^{D_c}(\cdot)$ is of quasi scalar type. The self-adjoint extension $D_c^{(1)} := D_c^* \upharpoonright \ker(\Gamma_0^{D_c})$ has the domain

$$\mathrm{dom}\,(D_c^{(1)}) = \{f \in W^{1,2}(\Delta_c, \mathbb{C}^2) : f_1(a) = f_1(b) = 0\}$$

while the extension $D_c^{(2)} := D_c^* \upharpoonright \ker(\Gamma_1^{D_c})$ has the domain

$$\mathrm{dom}\,(D_c^{(2)}) = \{f \in W^{1,2}(\Delta_c, \mathbb{C}^2) : f_2(a) = f_2(b) = 0\}.$$

We consider the closed symmetric operator

$$S_c := \overline{D_c \otimes I_{\mathfrak{T}} + I_{\mathfrak{D}_c} \otimes T}$$

which is defined on $\mathfrak{K}_c := \mathfrak{D}_c \otimes \mathfrak{T} = L^2(\Delta_c, (\mathfrak{T} \oplus \mathfrak{T})^t)$. In the following we denote elements of of $\mathfrak{K}_c$ by $\vec{f}$. In particular, we use the notation

$$\vec{f} = \begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \end{pmatrix}, \quad \vec{f}_j \in L^2(\Delta_c, \mathfrak{T}), \quad j = 1, 2.$$

Let us construct the boundary triplet $\Pi_{S_c} = \{\mathcal{H}^{S_c}, \Gamma_0^{S_c}, \Gamma_1^{S_c}\}$ for S_c^*. Since the Weyl function $M^{D_c}(\cdot)$ is of quasi scalar type we follow Remark 4.10. To this end we introduce the subspaces $\mathcal{H}_1^{D_c} := \mathbb{C}$ and $\mathcal{H}_2^{D_c} = \mathbb{C}$. This yields $\mathcal{H}_1^{S_c} = \mathfrak{T}$ and $\mathcal{H}_2^{S_c} = \mathfrak{T}$ as well as

$$\mathcal{H}^{S_c} = \begin{array}{c} \mathcal{H}_1^{S_c} \\ \oplus \\ \mathcal{H}_2^{S_c} \end{array} = \begin{array}{c} \mathfrak{T} \\ \oplus \\ \mathfrak{T} \end{array}.$$

Furthermore, we have

$$\Gamma_0^{S_c}\vec{f} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\mathrm{Im}(m_1^{D_c}(i-T))}(\vec{f}_1(a) + \vec{f}_1(b)) \\ \sqrt{\mathrm{Im}(m_2^{D_c}(i-T))}(\vec{f}_1(a) - \vec{f}_1(b)) \end{pmatrix}$$

and

$$\Gamma_1^{S_c}\vec{f} = \frac{ic}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\mathrm{Im}(m_1^{D_c}(i-T))}} \left(\vec{f}_2(a) - \vec{f}_2(b) - \mathrm{Re}(m_1^{D_c}(i-T))(\vec{f}_1(a) + \vec{f}_1(b))\right) \\ \frac{1}{\sqrt{\mathrm{Im}(m_2^{D_c}(i-T))}} \left(\vec{f}_2(a) + \vec{f}_2(b) - \mathrm{Re}(m_2^{D_c}(i-T))(\vec{f}_1(a) - \vec{f}_1(b))\right) \end{pmatrix},$$

$\vec{f} \in \mathrm{dom}\,(D_c^* \otimes I_{\mathfrak{T}}) \cap \mathrm{dom}\,(I_{\mathfrak{D}_c} \otimes T)$. The Gamma field $\gamma^{S_c}(\cdot) : \mathcal{H}^{S_c} \longrightarrow \mathfrak{K}_c$ is computed by

$$(\gamma^{S_c}(z)\vec{\xi})(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\cos(k(z-T)(x-\nu))}{\cos(k(z-T)d)\sqrt{\mathrm{Im}(m_1^{D_c}(i-T))}} & -\frac{\sin(k(z-T)(x-\nu))}{\sin(k(z-T)d)\sqrt{\mathrm{Im}(m_2^{D_c}(i-T))}} \\ i\frac{k_1(z-T)\sin(k(z-T)(x-\nu))}{\cos(k(z-T)d)\sqrt{\mathrm{Im}(m_1^{D_c}(i-T))}} & i\frac{k_1(z-T)\cos(k(z-T)(x-\nu))}{\sin(k(z-T)d)\sqrt{\mathrm{Im}(m_2^{D_c}(i-T))}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad z \in \mathbb{C}_\pm\,.$$

The Weyl function $M^{S_c}(\cdot) : \mathcal{H}^{S_c} \longrightarrow \mathcal{H}^{S_c}$ is given by

$$M^{S_c}(z) = \begin{pmatrix} \frac{m_1^{D_c}(z-T) - \mathrm{Re}(m_1^{D_c}(i-T))}{\mathrm{Im}(m_1^{D_c}(i-T))} & 0 \\ 0 & \frac{m_2^{D_c}(z-T) - \mathrm{Re}(m_2^{D_c}(i-T))}{\mathrm{Im}(m_2^{D_c}(i-T))}, \end{pmatrix}, \qquad z \in \mathbb{C}_\pm.$$

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